Technical University Berlin, Germany
Institute of Mathematics
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Optimal execution in limit order book markets with call auctions

Diploma Thesis

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Eidesstattliche Erklärung

Die selbständige und eigenhändige Anfertigung versichere ich an Eides Statt.

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Zusammenfassung

Die vorliegende Arbeit entstand im Rahmen einer Stelle als studentische Hilfskraft am Quantitative Products Laboratory, einem Forschungsinstitut der Humboldt- und der Technischen Univerität Berlin gefördert durch die Deutsche Bank AG.
Es wird der Fall eines Großinvestors betrachtet, der innerhalb eines vorgegebenen Zeithorizonts $[0, T]$ eine Aktienposition erwerben (oder analog verkaufen) möchte, die einen signifikanten Anteil des täglichen Handelsvolumens des betrachteten Titels darstellt.
Im Vergleich zu einem Kleinanleger werden dabei jedoch Liquiditätskosten fällig. Deren Modellierung beeinflusst maßgeblich die schließlich gesuchte kostenminimale Kaufstrategie des Großinvestors.
Unser Modell ist dabei inspiriert durch die Arbeit von Obizhaeva und Wang [20]. Es basiert auf einem elektronischen Handelssystem wie Xetra, in dem Limit- und Marktorders gestellt werden können. Bei einer Kauf- bzw. Verkaufslimitorder gibt der jeweilige Marktteilnehmer die Anzahl von Aktien sowie den maximalen bzw. minimalen Preis pro Aktie an, den er zu zahlen bzw. zu akzeptieren bereit ist. Diese Limitorders werden dann im sogenannten Limitorderbuch (LOB) gesammelt und können durch Stellen einer unlimitierten Marktorder von anderen Marktteilnehmern benutzt werden. In unserem Modell vereinfachen wir das LOB indem wir davon ausgehen, dass es eine Blockform aufweist. D.h. zu jedem Angebots- und Nachfragepreis gibt es eine konstante Anzahl von Aktien. Stellt der Großinvestor nun eine Kaufmarktorder mit einer größeren Aktienanzahl als die Höhe des Blocks, so erhöht dies den besten Nachfragepreis im LOB. Durch die Blockform ist dieser Preiseinfluss linear in der Anzahl der Aktien der Marktorder. Ein Teil des Einfluss ist permanenter Natur; der restliche temporäre Preiseinfluss nimmt nach und nach ab, wobei dieses Abklingen exponentiell modelliert wird.

Es stellt sich die Frage, wie der Großinvestor seine Aktienposition über den betrachteten Zeithorizont in Marktorders stückelt, um den Erwartungswert der ihm entstehenden Kosten zu minimieren. Durch dynamische Programmierung ergibt sich dabei als optimale Kaufstrategie, zwei gleich große diskrete Trades in den Zeitpunkten 0 und T zu tätigen und die verbleibende Position gleichmäßig über $(0, T)$ zu verteilen. Wir weisen nach, dass diese Strategie zeitkonsistent ist. Das Charakteristische an ihr ist die Tatsache, dass der temporäre Einfluss auf einem konstanten Level gehalten wird. Diese Resultate werden sowohl für diskrete als auch stetige Marktmodelle dargestellt. Berücksichtigt man bei der Optimierung zusätzlich auch die Varianz der Kosten durch Einführung eines Risikoaversionskoeffizientens, wird bei der optimalen Strategie der Handel zeitlich nach vorne verlagert. Wie bereits angedeutet sind diese Ergebnisse weitestgehend der Arbeit von Obizhaeva und Wang [20] entnommen und werden in verbesserter Art und Weise dargestellt.

Auf dieser Basis werden dann verschiedene Erweiterungsmöglichkeiten und Verfeinerungen des Modells von uns diskutiert: So führen wir zum Beispiel einen zusätzlichen linearen temporären Preiseinfluss ein, der lediglich instantan wirkt und in dieser Form in der Arbeit von Almgren und Chriss [2] zu finden ist. Die sich ergebende optimale Kaufstrategie hat dann einen U-förmigen Verlauf. Des Weiteren ersetzen wir konstante

Modellparameter wie die Blockhöhe durch ihre deterministischen Tagesverläufe.

Außerdem wird untersucht, wie sich sogenannte Auktionen in das Modell einbetten lassen. Dies wird relevant, sobald unser Zeithorizont mehrere Handelstage berührt. Dabei verstehen wir hier unter einer Auktion den an vielen Börsenplätzen zu findenden Mechanismus zur Feststellung eines Eröffnungs- bzw. Schlusskurses. Dazu wird der Handel für einige Minuten sowohl morgens als auch abends ausgesetzt. Vorhandene und während der Auktion neu eingegebene Orders werden gesammelt, um nach einem bestimmten Auktionsmechanismus den Preis pro Aktie zu ermitteln, aus dem das maximale Handelsvolumen resultiert. Dabei haben die von uns berechneten, je nach Modellvariante unterschiedlichen optimalen Kaufstrategien folgende Gemeinsamkeiten: Nicht nur in den Zeitpunkten 0 und T, sondern auch auf den Auktionen selbst und den unmittelbar angrenzenden Handelszeitpunkten sind diskrete Trades zu tätigen. Außerdem wird zwischen den Auktionen, d.h. während des kontinuierlichen Handels, weiterhin mit konstanter Intensität gehandelt.

Schließlich beleuchten wir, was passiert, wenn wir die Blockform des LOB durch beliebige positive, stetige Funktionen ersetzen. Erstaunlicherweise ändert dies die optimale Strategie kaum. Lediglich die beiden diskreten Trades in 0 und T sind nicht mehr gleich groß.

Abstract

The following work has been written during a student occupation at the Quantitative Products Laboratory, which is a research institute of the Humboldt- and the Technical University Berlin sponsored by the Deutsche Bank AG.
We consider the case of an institutional investor who wants to purchase (or analogously liquidate) a given position of one asset representing a significant portion of daily trading volume in a fixed time period $[0, T]$.
In comparison to a small investor this entails liquidity cost whose modelling is of crucial importance when looking for the minimal cost buying strategy for the institutional investor. Our model is inspired by the paper of Obizhaeva and Wang [20] and is based on an electronic trading system as Xetra. Trading occurs by stating limit and market orders. When a market participant states a limit buy or sell order he indicates the number of shares as well as the maximal respectively minimal price per share he is willing to pay or rather accept. These limit orders are collected in the so-called limit order book (LOB) and can be used by others by quoting a market order. We simplify the LOB by assuming it having a block shape, which means that there is a constant number of shares for each bid and ask price in the book. If the institutional investor executes a market order with a larger number of shares than the block height, this will increase the best ask price in the LOB. Due to the block shape of the LOB, this price impact is linear in the number of shares of the market order. One component of the impact is permanent and the decay of the temporary component is modelled exponentially.

The question arises how to split the position of the institutional investor into market orders in order to minimise the expected cost. Using dynamic programming, the optimal buying strategy is to put two equally sized discrete trades at the times 0 and T and to spread the remaining shares uniformly over $(0, T)$. We see that this strategy is time-consistent and that it is characterised by a constant level of temporary impact. We present these results for discrete as well as continuous time market models. When not only the expectation, but also the variance of the cost are considered for the optimisation by introducing a risk aversion coefficient, the optimal strategy changes in the sense that trading is shifted forward in time. As already indicated, most of these results are taken from the paper of Obizhaeva and Wang [20] and are going to be displayed in an improved manner.

On this basis, we proceed with discussing various extensions and refinements of the model: For instance we introduce an additional linear temporary impact which only acts instantaneously and can be found in the work of Almgren and Chriss [2]. The resulting optimal buying strategy then has a U-shape. Moreover, we replace constant model parameters like the block height by their deterministic daily developing.

Additionally we examine how so-called call auctions can be incorporated into the model. This is relevant as soon as our time horizon spans several trading days. We mean by an auction the trading mechanism which can be found at most of the market places and that serves for the determination of the opening and the closing price of the asset. To

do so, trading is frozen for a few minutes each morning and evening in order to assess the price per share that results in a maximum traded volume. Hence, the auctions disconnect the continuous trading that we modelled before. We finally formulate different ideas to model these auctions. The resulting optimal buying strategies all have discrete trades not only at time 0 and T, but also on the auctions and directly next to them. Their buying intensity is again constant during continuous trading.

Lastly, we shed light on what happens when the block shape of the LOB is replaced by a positive, continuous function. Astonishingly, this changes the optimal strategy just slightly. Only the discrete trades at 0 and T do not have equal size anymore.

Contents

1 Introduction

Let us assume that an institutional investor like an insurance company or a pension fund comes to a trader with the order to purchase a big package of shares of one asset. The investor requires a deadline on the scale of some days for the order to be fulfilled. Typically, the considered position constitutes a significant fraction of the stock's trading volume. Therefore, an instantaneous acquisition is not advisable, since it would cause a large adverse price impact. Thus, the order should be split into several parts with the intention to achieve minimal expectation and variance of the cost thereby incurred. In doing so, it is critical to respect the fact that buying raises the stock price, which inevitably affects the price for the shares still to be purchased afterwards. Hence, the modelling of this price impact is of crucial importance. It consists of a temporary and a permanent component as e.g. described in the empirical paper of Holthausen [13]. The temporary part compensates the sellers for providing short-term liquidity and the permanent one is due to the fact that the sellers presume the buyer to possess asymmetric information like insider information. In the following we use the same price impact as in the paper of Obizhaeva and Wang [20] because the temporary price impact is modelled more dynamic in comparison to other work on this field. Another advantage of the model of Obizhaeva and Wang is that discrete and not only continuous trading is allowed.
Although we only discuss the purchase of shares, the dual problem of selling shares can be solved analogously.

We are going to model the problem described above in a market where trading is processed solely using a so-called **limit order book (LOB)**. An example for an electronic trading system with a LOB is the Xetra system of the Deutsche Börse, which has operated since 1997 and is successively replacing classical floor trading. Market participants can submit market and limit orders in order to trade with each other. A **limit order** generally consists of three pieces of information that the investor has to specify: how many shares he is willing to buy or rather sell at a fixed maximal or minimal execution price respectively and the expiry date. These limit buy and sell orders are then collected in the LOB and can be seen by all market participants. An example of this can be found in Figure 1. The displayed highest bid price is smaller than the lowest ask price at all times, since matching limit orders in the LOB are executed immediately. We say that two limit orders are matching when one is a sell and the other a buy order with the maximal buy price being higher or equal to the minimal sell price. If the sell order is registered with $x_s \in \mathbb{N}$ shares and the limit buy order with $x_b \in \mathbb{N}$, the minimum of these two share amounts can finally be traded.

With a **market order** the investor is only stating the number of shares he is willing to buy or sell. Market orders are executed right away by being assigned to the best limit sell or buy orders by the system, which are then not recorded anymore in the LOB. In short, limit orders are listed with their prices in the LOB and wait for other counterpart limit and mainly market orders to come. Thus a limit order achieves a better price compared to a market order, but bears the risk of being not or only partially executed. In this respect, a market order consumes and a limit order provides liquidity. A market sell order corresponds to a limit sell order with a limit price equal to zero.

The following priority rules are used in a general LOB market system to decide which orders are executed: If there are limit orders having the same price, priority will be given to the one which was set earlier. This is made clear in Figure 1 by means of the two limit buy orders with price 967. For example a limit sell order arriving at 13.48h with execution price 967 for example affects the first three limit buy orders. More precisely 500 shares are sold at 967.5, 800 at 967 and the remaining 200 shares at 967 as well, such that the more recent order with price 967 only comprises 743 shares in the LOB afterwards.
If the explained limit sell order would had spanned 2500, instead of 1500 shares with price 967, the three best limit buy orders would all have been executed completely. The remaining $2500 - 500 - 800 - 943 = 257$ shares would stay in the LOB as limit sell order and 967 is the new best ask price.
In general, the orders listed in the LOB are anonymous. Other attributes and functionalities of the LOB can vary depending on the market place, for instance the number of ticks of the LOB that are publicly visible.

Buy Orders				**Sell Orders**		
Buy vol.		25,743		Sell vol.		32,632
Number of orders		13		Number of orders		10
Avg. size		1,980		Avg. size		3,263
1	**500**	**967.5**	**-**	**968.5**	**158**	**1**
13:47	500	967.5		968.5	158	13:45
13:46	800	967.0		969.5	370	13:38
13:47	943	967.0		970.0	200	13:48
13:45	262	966.5		970.5	404	13:33
13:07	227	965.0		975.5	3,000	13:44
12:49	600	964.5		986.0	1,000	13:19
12:21	657	964.0		987.0	2,000	12:33
10:22	75	956.5		998.0	10,000	08:34
13:43	3,280	956.0		1000.0	15,000	07:55
09:33	899	954.0		1200.0	500	07:53
12:11	500	950.5				
10:08	2,000	950.0				
08:02	15,000	910.0				

Figure 1: Fictitious example of a limit order book.

The proceeding work is organised as follows: In Chapter 2, the model of the LOB is introduced. In Chapter 3 and 4 we derive the optimal buying strategy, where discrete respectively continuous trading is allowed. Chapter 5 presents a possibility to extend the LOB model by allowing time-dependent market depth, etc. In Chapter 6 to 8, various ideas to include call auctions in the setting are discussed. Finally, Chapter 9 presents another extension of the model: More general forms of the LOB are considered.

2 Dynamics of the limit order book

Before we introduce our mathematical model of a LOB, which was established by Obizhaeva and Wang in their paper [20], we want to state this interesting definition of **liquidity** of a market by Kyle [16] as motivation. He distinguishes three aspects of liquidity:

- **Spread:**
  By tightness Kyle means the cost which arises from buying and selling a position back-to-back. This aspect refers to the spread. The bid-ask spread is the difference between the lowest selling and the highest buying price in a market.
- **Depth:**
  The market depth is the size of the order which is necessary to change the best bid or ask price respectively.
- **Resiliency:**
  Resiliency is the ability of a market to recover from a price shock (see below).

The market model with one risky asset that will be introduced in the sequel, comprises all of these three liquidity aspects. For this purpose let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,\infty)})$ be the probability space of our model. It is equipped with the filtration generated by a one-dimensional Brownian motion W. Since in our context the market model is only considered on a time horizon with the magnitude of a few days, it is sufficient to use the simple Bachelier model for the **equilibrium price process** $(S_t)_{t\in[0,\infty)}$ of the asset:

$$S_t = S_0 + \sigma W_t,$$

where the constant σ constitutes the volatility of the asset. We do not make use of a drift term and negative values might occur.

In order to describe how the LOB reacts to a sequence of purchases of shares, we first have a look at the basic state of the LOB at time $t = 0$. The best ask price modelled by the left-continuous process A_t is always larger than the best bid price B_t. For the time being we neglect the impact that our purchase of the X_0 shares will have on the best ask and set $A_t = S_t + \frac{z}{2}$ and $B_t = S_t - \frac{z}{2}$, respectively, where the positive constant z is the so-called **bid-ask spread**[1]. Typically, it holds that the higher the liquidity in the market, the lower the spread. In this sense we could call S the mid-quote price. In Figure 1 we thus have $A_0 = 968.5$, $B_0 = 967.5$ and $z = 1$.

As displayed in Figure 2, we assume in our simple model that the LOB has a block form $qI_{\{y\leq B_t, y\geq A_t\}}$ where I denotes the indicator function.
That means that to every ask and bid price at time $t \in [0,\infty)$ there is a constant number $q \in \mathbb{R}_+$ of shares listed in the LOB. Therefore, we call q the **market depth**. It is the second measure of liquidity that we mentioned above. As implied by Figure 2, bid and ask prices as well as the spread z are measured in so-called ticks. The tick-size

[1] Instead of a constant bid-ask spread we could also take a Bachelier model for z. This would not change our later results.

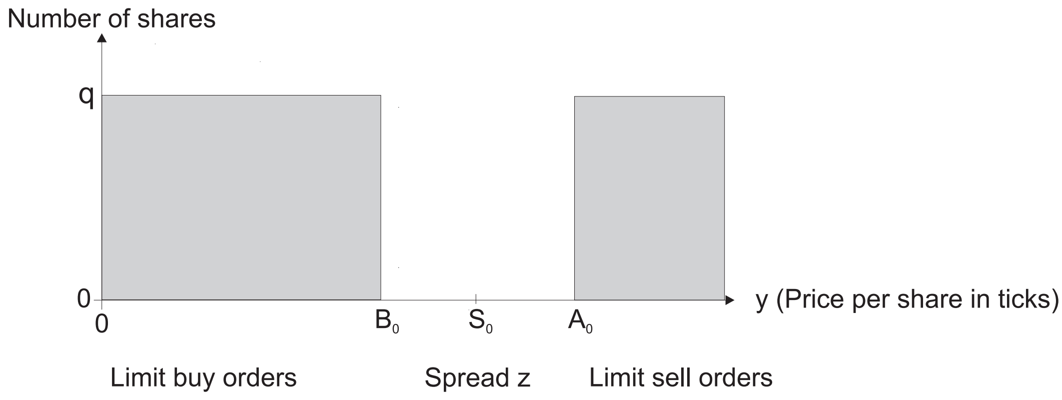


Figure 2: Block form of the LOB in $t = 0$.

is the smallest possible price change of the considered asset at the market place under consideration (e.g. Euro cent).

The assumption of this block shape is of course a significant simplification of the true market depth, which in reality is not constant. Rather it depends on the time as well as the price. However, we will ease this simplification later in Chapter 5 and 9.
A typical evolution of the cumulative, i.e. summed up market depth can be found in Figure 3, which is taken from the master thesis of Steinmann [23]. It has to be interpreted as follows: A volume of v^* shares at time t^* and a price p^* smaller than the best bid B_{t^*} means that there are limit buy orders over v^* shares in the book at time t^* with a limit price between p^* and B_{t^*}. Therefore, the volume in Figure 3 is for a fixed time t increasing for prices p bigger than A_t and decreasing for prices smaller than B_t.

Since we focus on the purchase of shares by using market buy orders which consume limit sell orders, the consideration of the left part of the LOB, containing the limit buy orders, can be neglected for the time being.

When a market buy order over $x_0 \in \mathbb{N}$ shares is executed at time $t = 0$, it is matched with the lowest limit sell orders from the LOB. Therefore, the best ask price increases to A_{0+}:

$$(A_{0+} - A_0)\, q = x_0 \qquad\qquad \Leftrightarrow \qquad\qquad A_{0+} = A_0 + \frac{1}{q} x_0 = \left(S_0 + \frac{z}{2}\right) + \frac{1}{q} x_0.$$

The **average buying price** of the market order is

$$\overline{P}_0 = \frac{A_0 + A_{0+}}{2} = A_0 + \frac{1}{2q} x_0 = \left(S_0 + \frac{z}{2}\right) + \frac{1}{2q} x_0.$$

The price increase caused by the order is $\frac{1}{q} x_0$. Due to the block form of the LOB it is linear in x_0 and the higher, the smaller the market depth q is.

For the time being, we assume no other market orders, except of x_0, occur. Then the question arises how the explained price increase is offset by newly arriving limit sell

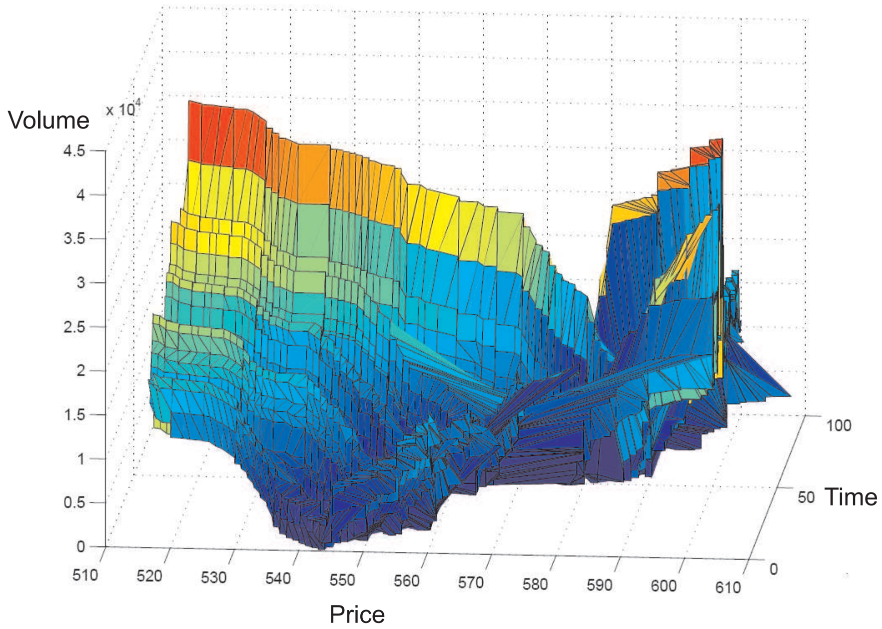


Figure 3: Actual form of the LOB which is depending on the price and time on the basis of the Givaudan stock from April 1 to 16, 2002 at the Swiss Stock Exchange. Listed is the cumulative volume!

orders. We call the part of the price impact which persists the **permanent price impact**. It is measured by the constant λ and is due to the information we reveal to the market. The other part of the price impact which will be compensated for is called **temporary price impact** with the associated constant κ. That is we obtain

$$\frac{1}{q}x_0 = \lambda x_0 + \kappa x_0,$$

where we calibrate the constant λ such that $0 \leq \lambda < \frac{1}{q}$ (e.g. $\lambda = \frac{1}{2q}$) and

$$\kappa := \frac{1}{q} - \lambda > 0. \tag{1}$$

The distinction between permanent and temporary price impact is undertaken by several authors. But the question arises how long the temporary part exists. In the work of Almgren and Chriss [2] it is only existent at the considered trading point in time and disappears immediately afterwards. In contrast, Holthausen, Leftwich and Mayers assume in [13] that the temporary impact stays constant during the remaining trading day and is totally gone on the next. Both alternatives are unrealistic in the context of a LOB market. The **exponential decay** in our model appears more reasonable:
The temporary price impact at time t caused by the purchase x_0 at time zero amounts to

$$\kappa x_0 e^{-\rho t}. \tag{2}$$

Other market participants quote bit by bit new limit sell orders to fill in the gap κx_0 and to eliminate this temporary pricing error. One can interpret the positive constant ρ as a measure for the price recovery which we will call **resiliency** of the LOB. There is no unanimous opinion on the size of this constant. According to Dong, Kempf and Yadav [8] it is governed by parameters like transaction frequency, relative tick size, average transaction size and realized spread. Depending on the market place, the associated **half-life** ϑ with $e^{-\rho\vartheta} = \frac{1}{2}$ should lie in the range of a few minutes or even seconds. The sample of heavily-traded New York Stock Exchange stocks used in [8] has e.g. a resiliency of 60 percent over a one-minute horizon. Through the competitive actions of other market participants only 40 percent of the temporary pricing error is left after one minute.

Let us now introduce two more processes: The so-called intrinsic price $\hat{S}_t$ is made up by the fair value S_t plus the permanent price impact which accumulated during $[0, t)$. In our case we have $\hat{S}_t = S_t + \lambda x_0$.
We define the temporary impact by $D_t := A_t - (\hat{S}_t + \frac{z}{2})$. It is the deviation between the actual best ask price and the intrinsic best ask price at time t. Both processes are left-continuous. We use them in Figure 4 to summarise what we have explained so far.

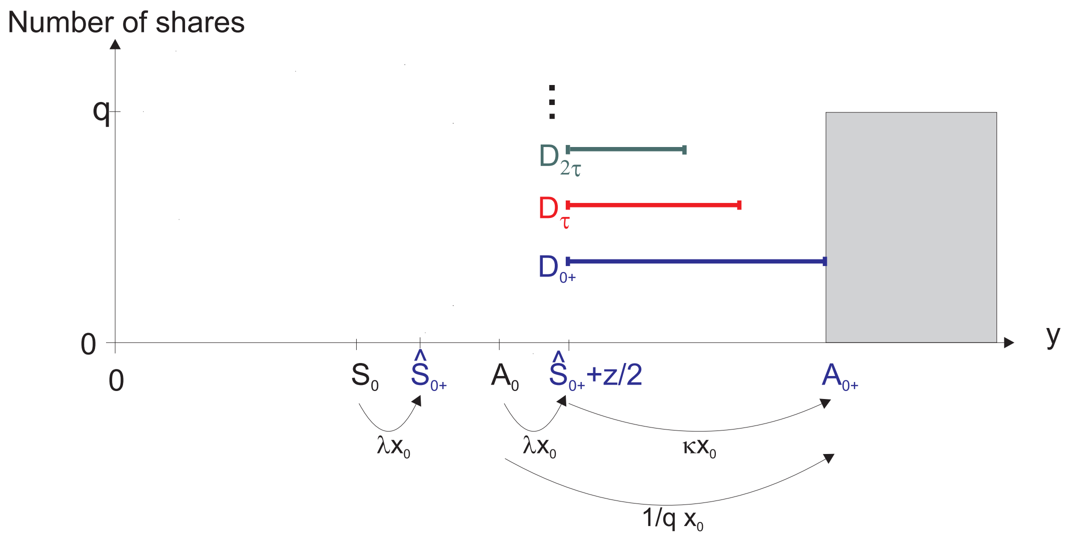


Figure 4: Schematic description of the reaction of the LOB to the purchase in $t = 0$. Thereby τ is the time which elapses between two trading points in time.

We get a totally analogous behaviour of the LOB when we have several purchases $x_0, ..., x_n$ at the times $t_0, ..., t_n < t$:
The intrinsic price results from

$$\hat{S}_t = S_t + \lambda \sum_{i=0}^{n} x_i$$

and the best ask price is

$$\begin{aligned} A_t &= \left(S_t + \frac{z}{2}\right) + \lambda \left(X_0 - X_t\right) + D_t \qquad (3) \\ &= \left(S_t + \frac{z}{2}\right) + \lambda \left(X_0 - X_t\right) + \kappa \sum_{i=0}^{n} x_i e^{-\rho(t-t_i)}. \end{aligned}$$

Thus, the process D accounts for the whole temporary impact accumulated until time t taking into account the exponential decay. With X_t we refer to the number of shares we still have to buy in the time interval $[t, T]$ where T denotes the end of our trading time interval, which is fixed by the investor. Both processes D and X are left-continuous. After having presented our price dynamics, we can now start deriving optimal trading strategies.

3 Optimal trading strategy in discrete time

A trader gets the job to buy a package of $X_0 \in \mathbb{N}$ shares of one kind in a fixed period of time $[0, T]$. As explained in the introduction, we assume X_0 to be a substantial amount of shares. Therefore the package has to be split up. In this chapter, we look for the left-continuous, decreasing process $(X_t)_{t\in[0,T]}$ with $X_T = 0$ such that the trader has minimal cost. We are not accounting for stock-exchange fees or transaction cost.

In order to approach the problem stated above and to compare the following results with the ones of other authors, we firstly impose the restriction that trading is only possible at fixed discrete trading times. Therefore, we consider the equidistant decomposition

$$0 = t_0 < t_1 < ... < t_N = T$$

of the interval $[0, T]$ with constant

$$\tau := t_{n+1} - t_n = \frac{T}{N}.$$

Thus by choosing N appropriately large, the trading points in time may lie arbitrarily near to each other. A **trading strategy**, or more precisely a buying strategy, is now given by its deterministic coefficients $x_0, ..., x_N \in \mathbb{R}_{\geq 0}$ with

$$\sum_{n=0}^{N} x_n = X_0. \tag{4}$$

We have $x_n = X_{t_n} - X_{t_{n+1}}$ for $n = 0, ..., N-1$ and accordingly $X_{t_N} = X_T = X_0 - \sum_{n=0}^{N-1} x_n$. Therefore, one sets $x_N = X_T$ in order to assure (4).
Just as in [2] and [3] by Almgren and Chriss, we look for a **static strategy** whose coefficients x_n depend only on the information available at time $t = 0$. In a **dynamic strategy**, however, the coefficients can depend on events from $\mathcal{F}_{t_{n-1}}$.

We will say that a trading strategy is **optimal** if the expected total cost of the purchase is minimised. This means that the buyer is risk-neutral. In the following the adapted process C_t refers to the expected cost under the optimal strategy that will occur in the interval $[t, T]$. The expected total cost from time $t = 0$ onwards is given by

$$C_0 := \min_{\left\{x_0,...,x_N \in \mathbb{R} \mid \sum_{n=0}^{N} x_n = X_0\right\}} \mathbb{E}\left[\sum_{n=0}^{N} \overline{P}_{t_n} x_n\right]. \tag{5}$$

We will see in Proposition 1 that our optimal strategy $\{x_0, .., x_N\}$ is as desired positive. Therefore

$$C_0 = \min_{\left\{x_0,...,x_N \in \mathbb{R}_{\geq 0} \mid \sum_{n=0}^{N} x_n = X_0\right\}} \mathbb{E}\left[\sum_{n=0}^{N} \overline{P}_{t_n} x_n\right].$$

We get the average buying price at time t_n via equation (3) as

$$\overline{P}_{t_n} = \frac{A_{t_n} + A_{t_n+}}{2} = A_{t_n} + \frac{1}{2q} x_n = \left(S_{t_n} + \frac{z}{2}\right) + \lambda \left(X_0 - X_{t_n}\right) + D_{t_n} + \frac{1}{2q} x_n. \tag{6}$$

The accumulated temporary price impact is given by the dynamic:

$$D_0 = 0, \qquad D_{t_{n+1}} = \left(D_{t_n} + \kappa x_n\right) e^{-\rho\tau}. \tag{7}$$

That is the increase of the temporary impact due to the purchase at time t_n, as well as the decrease caused by the flow of newly arriving limit sell orders, are accounted for.

In the following proposition we state the optimal trading strategy and the process C_t in our model explicitly.

Proposition 1. *(Optimal trading strategy in discrete time)*
The expected cost under the optimal strategy is

$$C_{t_n} = \left(S_{t_n} + \frac{z}{2}\right) X_{t_n} + \lambda X_0 X_{t_n} + \left[\alpha_n X_{t_n}^2 + \beta_n X_{t_n} D_{t_n} + \gamma_n D_{t_n}^2\right] \tag{8}$$

with the following sequences α_n, β_n and γ_n for $n = 0, ..., N$:

$$\begin{aligned} \alpha_n &= \frac{(1+e^{\rho\tau}) - q\lambda\left[(N-n)(e^{\rho\tau}-1) + 2(1+e^{\rho\tau})\right]}{2q\left[(N-n)(e^{\rho\tau}-1) + (1+e^{\rho\tau})\right]} \\ \beta_n &= \frac{1+e^{\rho\tau}}{\left[(N-n)(e^{\rho\tau}-1) + (1+e^{\rho\tau})\right]} \\ \gamma_n &= \frac{(N-n)\left(1-e^{\rho\tau}\right)}{2\kappa\left[(N-n)(e^{\rho\tau}-1) + (1+e^{\rho\tau})\right]}. \end{aligned} \tag{9}$$

The associated optimal trading strategy is given by

$$\begin{aligned} x_0 &= x_N = X_0 \frac{1}{(N-1)\left(1-e^{-\rho\tau}\right) + 2} \quad \text{and} \\ x_n &= \frac{X_0 - 2x_0}{N-1} = X_0 \frac{1-e^{-\rho\tau}}{(N-1)\left(1-e^{-\rho\tau}\right)+2} \quad \text{for } n = 1, ..., N-1. \end{aligned} \tag{10}$$

In particular, we have $0 < x_i < \frac{1}{2}X_0$ for $i \in \{0, N\}$ and $0 < x_n < \frac{1}{2}X_{t_n}$ for $n = 1, ..., N-1$.

The main part of the proof of the proposition above is Lemma 4. It is taken from the Obizhaeva and Wang paper [20]. But in comparison to Obizhaeva and Wang, we additionally managed to specify the sequences α_n, β_n and γ_n appearing in the optimal strategy and in the cost term C_{t_n} explicitly instead of only recursively. This reduces e.g. the time to calculate the optimal strategy and particularly enables us to show its positivity.

3.1 Interpretation of Proposition 1

Before we turn to the proof of Proposition 1 in Subsection 3.2, we want to analyse the optimal strategy and cost term.

Figure 5 illustrates the result of Proposition 1. It shows the optimal trading strategies $\{x_0, ..., x_N\}$ for $N = 10$ and 50, respectively. These are characterised by two peaks

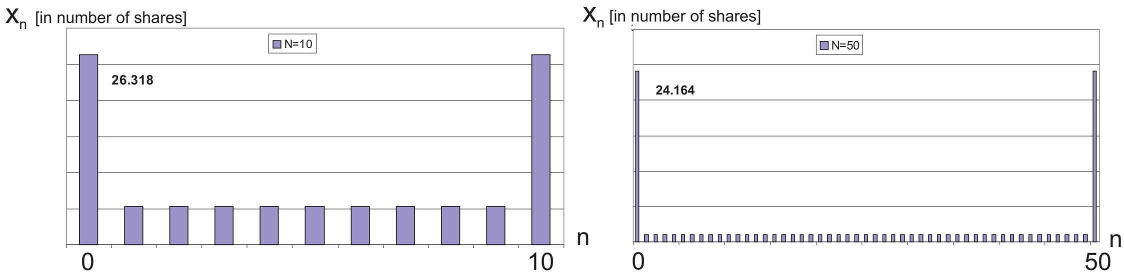


Figure 5: Optimal trading strategy in discrete time for $N = 10$ and $N = 50$. We set $\rho = 2.31$. The values for the remaining parameters are given in Appendix A.1.

x_0 and x_N of equal size as well as an evenly spreading of the remaining shares over the trading times $t_1, ..., t_{N-1}$. Here, x_n refers to the number of shares to be traded at time t_n as a market buy order. If one intended to implement the strategy given in Proposition 1 in practice, it would be advisable to make use of limit orders in order to buy the x_n shares in the time interval $[t_n, t_{n+1})$. Otherwise the spread has to be crossed for each single trade. How this problem can be approached is presented e.g. by Nevmyvaka et al. in [19].

The trading profile depicted in Figure 5 is intuitive and can be explained as follows: The initial discrete trade of x_0 shares uses the best limit sell orders. This has the positive effect of resiliency–as stated in the work of Bias, Hillion and Spatt [6], new limit sell orders are quickly attracted and are reducing the growing spread. The sellers are willing to undercut the existing best ask price, since this increases the probability that their limit sell orders are executed. But the initial purchase x_0 should not be too big as this would unnecessarily increase the average price per share of the succeeding orders.
The following constant trading matches the newly attracted limit sell orders. Characteristic for the optimal strategy is that the traded number of shares per trading period is chosen such that the deviation between the actual and the intrinsic best ask price stays on a constant level. This will be shown in Lemma 6 and is illustrated in Figure 6. Therefore, the flow of incoming limit sell orders is constant over $(0, T)$.
The trade x_N at the end of the time horizon $[0, T]$ has a high price impact. But since the purchase is finished anyway, this is only relevant for the last trade itself. A more detailed discussion of the nature of the optimal strategy can be found in Subsection 4.1 where the optimal strategy in continuous trading time will be analysed.

The structure of C_0 as appearing in (8) is notable. For example, we have

$$C_0 = \left(S_0 + \frac{z}{2} \right) X_0 + (\lambda + \alpha_0) X_0^2.$$

The expected cost under the optimal strategy to be assessed per share $\frac{C_0}{X_0}$ are linear in X_0.

Let us now analyse how C_0 depends on the time horizon T when keeping τ constant.

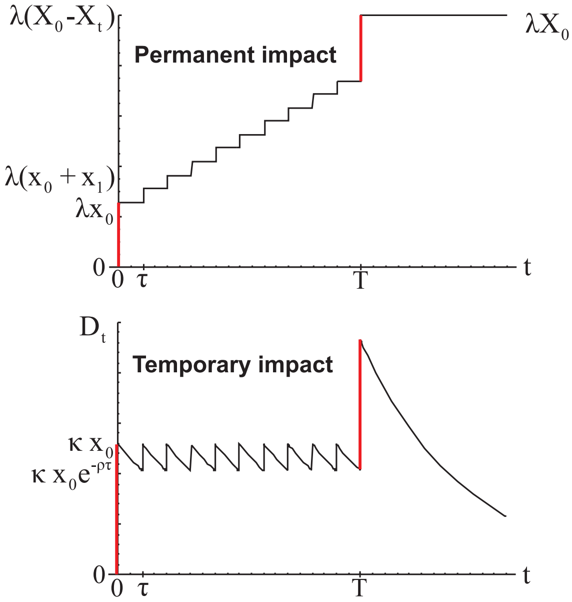


Figure 6: The two diagrams show the permanent and the temporary impact respectively belonging to the optimal strategy illustrated in the first diagram of Figure 5.

According to the form of α_0 from (9) we have

$$C_0(N) = \left(S_0 + \frac{z}{2}\right) X_0 + \lambda X_0^2 + X_0^2 \frac{(1 + e^{\rho\tau}) - q\lambda\left[N(e^{\rho\tau} - 1) + 2\,(1 + e^{\rho\tau})\right]}{2q\left[N\,(e^{\rho\tau} - 1) + (1 + e^{\rho\tau})\right]},$$

which is decreasing in N. Since α_0 converges to $-\frac{\lambda}{2}$ for $N \to \infty$ we get

$$\lim_{N\to\infty} C_0(N) = \left(S_0 + \frac{z}{2}\right) X_0 + \frac{\lambda}{2} X_0^2 \quad \text{and} \quad C_0(0) = \left(S_0 + \frac{z}{2}\right) X_0 + \frac{1}{2q} X_0^2.$$

Hence, $\lim_{N\to\infty} C_0(N) < C_0(0)$ for $\lambda < \frac{1}{q}$. This is illustrated in Figure 7. If we set $\lambda = \frac{1}{q}$, i.e. all impact is permanent, C_0 would be constant in N with $C_0(N) \equiv C_0(0) = \left(S_0 + \frac{z}{2}\right) X_0 + \frac{1}{2q} X_0^2$, since α_0 equals $-\frac{1}{2q} X_0^2$.

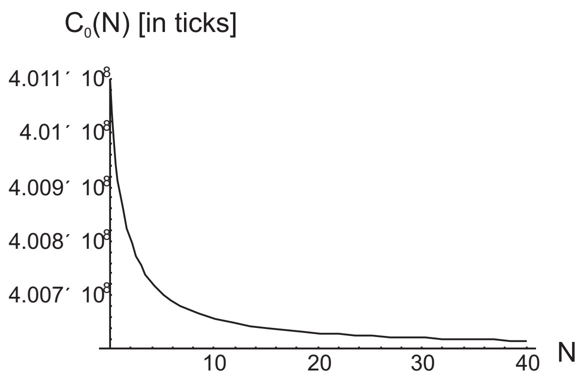


Figure 7: Plot of the cost function $C_0(N)$ for $\tau = \frac{1}{10}$ and $\rho = 20$. All other parameters can be found in the appendix.

3.2 Proof of Proposition 1

Let us now turn to the proof of Proposition 1. The main part of it is Lemma 4. It can be found in similar form in the Obizhaeva and Wang paper [20]. But in order to express the backward recursions occurring in Lemma 4 explicitly and to formally prove the existence of the optimal strategy, we need the following two auxiliary lemmata:

Lemma 2. *(Explicit formulas for the auxiliary sequences)*
We consider the sequences defined by the following backward recursions:

$$\begin{aligned}
\alpha_N = \frac{1}{2q} - \lambda \quad \textit{and} \quad \alpha_n &= \alpha_{n+1} - \frac{1}{4}\delta_{n+1}\epsilon_{n+1}^2 \\
\beta_N = 1 \quad \textit{and} \quad \beta_n &= \beta_{n+1}e^{-\rho\tau} + \frac{1}{2}\delta_{n+1}\epsilon_{n+1}\phi_{n+1} \\
\gamma_N = 0 \quad \textit{and} \quad \gamma_n &= \gamma_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}\phi_{n+1}^2
\end{aligned} \tag{11}$$

Thereby δ, ϵ and ϕ are defined as

$$\begin{aligned}
\delta_n &:= \left(\frac{1}{2q} + \alpha_n - \beta_n\kappa e^{-\rho\tau} + \gamma_n\kappa^2 e^{-2\rho\tau}\right)^{-1} \\
\epsilon_n &:= \lambda + 2\alpha_n - \beta_n\kappa e^{-\rho\tau} \\
\phi_n &:= 1 - \beta_n e^{-\rho\tau} + 2\gamma_n\kappa e^{-2\rho\tau}
\end{aligned} \tag{12}$$

Then we can write these six sequences explicitly for $n = 0, ..., N$ as

$$\begin{aligned}
\alpha_n &= \frac{(1+e^{\rho\tau}) - q\lambda\left[(N-n)(e^{\rho\tau}-1) + 2(1+e^{\rho\tau})\right]}{2q\left[(N-n)(e^{\rho\tau}-1) + (1+e^{\rho\tau})\right]} \\
\beta_n &= \frac{1+e^{\rho\tau}}{\left[(N-n)(e^{\rho\tau}-1) + (1+e^{\rho\tau})\right]} \\
\gamma_n &= \frac{(N-n)(1-e^{\rho\tau})}{2\kappa\left[(N-n)(e^{\rho\tau}-1) + (1+e^{\rho\tau})\right]}
\end{aligned} \tag{13}$$

$$\begin{aligned}
\delta_n &= \frac{2e^{2\rho\tau}\left[(N-n)(e^{\rho\tau}-1) + (1+e^{\rho\tau})\right]}{\kappa\left[(N-n)(1-e^{2\rho\tau}) + (N-n+2)(e^{3\rho\tau}-e^{\rho\tau})\right]} \\
\epsilon_n &= \frac{\kappa\left(e^{\rho\tau}-e^{-\rho\tau}\right)}{\left[(N-n)(e^{\rho\tau}-1) + (1+e^{\rho\tau})\right]} \\
\phi_n &= \frac{(N-n+1)(e^{\rho\tau}-e^{-\rho\tau}) - (N-n)(1-e^{-2\rho\tau})}{\left[(N-n)(e^{\rho\tau}-1) + (1+e^{\rho\tau})\right]}
\end{aligned} \tag{14}$$

Proof: We show by backward induction that the explicit formulae given in (13) for α, β and γ follow from (11) and (12). For $n = N$ it emanates from (13) as desired

$$\alpha_N = \frac{(1+e^{\rho\tau}) - 2q\lambda(1+e^{\rho\tau})}{2q(1+e^{\rho\tau})} = \frac{1}{2q} - \lambda, \quad \beta_N = 1 \quad \text{and} \quad \gamma_N = 0.$$

As induction hypothesis we substitute our expressions for α_{n+1}, β_{n+1} and γ_{n+1} from (13) to show (14):

$$
\begin{aligned}
\delta_{n+1}^{-1} &:= \frac{1}{2q} + \alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau} + \gamma_{n+1}\kappa^2 e^{-2\rho\tau} \\
&= \frac{\frac{1}{q}e^{2\rho\tau}\left[(N-n+1)e^{\rho\tau} - (N-n-3)\right] - \lambda e^{2\rho\tau}\left[(N-n+1)e^{\rho\tau} - (N-n-3)\right]}{2e^{2\rho\tau}\left[(N-n-1)\left(e^{\rho\tau}-1\right) + \left(1+e^{\rho\tau}\right)\right]} \\
&\quad + \frac{-2\kappa\left(e^{\rho\tau}+e^{2\rho\tau}\right) + \kappa(N-n-1)\left(1-e^{\rho\tau}\right)}{2e^{2\rho\tau}\left[(N-n-1)\left(e^{\rho\tau}-1\right) + \left(1+e^{\rho\tau}\right)\right]} \\
&= \frac{\kappa\left[(N-n-1)\left(1-e^{2\rho\tau}\right) + (N-n+1)\left(e^{3\rho\tau}-e^{\rho\tau}\right)\right]}{2e^{2\rho\tau}\left[(N-n-1)\left(e^{\rho\tau}-1\right) + \left(1+e^{\rho\tau}\right)\right]}
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{n+1} &:= \lambda + 2\alpha_{n+1} - \beta_{n+1}\kappa e^{-\rho\tau} \\
&= \frac{\frac{1}{q}\left(1+e^{\rho\tau}\right) - \lambda\left(1+e^{\rho\tau}\right) - \kappa\left(e^{-\rho\tau}+1\right)}{\left[(N-n-1)\left(e^{\rho\tau}-1\right) + \left(1+e^{\rho\tau}\right)\right]} \\
&= \frac{\kappa\left(e^{\rho\tau}-e^{-\rho\tau}\right)}{\left[(N-n-1)\left(e^{\rho\tau}-1\right) + \left(1+e^{\rho\tau}\right)\right]}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{n+1} &:= 1 - \beta_{n+1}e^{-\rho\tau} + 2\gamma_{n+1}\kappa e^{-2\rho\tau} \\
&= \frac{\left[(N-n-1)\left(e^{\rho\tau}-1\right) + \left(1+e^{\rho\tau}\right)\right] - \left(e^{-\rho\tau}+1\right) + (N-n-1)\left(e^{-2\rho\tau}-e^{-\rho\tau}\right)}{\left[(N-n-1)\left(e^{\rho\tau}-1\right) + \left(1+e^{\rho\tau}\right)\right]} \\
&= \frac{(N-n)\left(e^{\rho\tau}-e^{-\rho\tau}\right) - (N-n-1)\left(1-e^{-2\rho\tau}\right)}{\left[(N-n-1)\left(e^{\rho\tau}-1\right) + \left(1+e^{\rho\tau}\right)\right]}
\end{aligned}
$$

Putting these terms into the three equations in (11) we get the desired result (13) by another long calculation. □

Lemma 3. *(δ is strictly positive)*
For the sequence δ as given in Lemma 2 we have:

$$\delta_n > 0 \quad \text{for}\, n = 1, ..., N.$$

Proof: Because of the constants ρ and τ being strictly positive, the numerator of δ_n from (14) for $n = 1, ..., N$ is strictly positive. Since $\kappa > 0$ as well, we only have to consider the term

$$
\begin{aligned}
& (N-n)\left(1-e^{2\rho\tau}\right) + (N-n+2)\left(e^{3\rho\tau}-e^{\rho\tau}\right) \\
= \; & (N-n)\left(1-e^{\rho\tau}-e^{2\rho\tau}+e^{3\rho\tau}\right) + 2\left(e^{3\rho\tau}-e^{\rho\tau}\right)
\end{aligned}
$$

of the denominator. It is greater than zero because the function

$$h(x) := 1 - e^x - e^{2x} + e^{3x}$$

is strictly positive for $x = \rho\tau > 0$. □

We can now state our central lemma within the proof of Proposition 1. Its proof is a backward-induction which is done in detail here, since much of our later results will have the same proof structure.

Lemma 4. *(Dynamic programming)*
The expected cost under the optimal strategy is

$$C_{t_n} = \left(S_{t_n} + \frac{z}{2}\right) X_{t_n} + \lambda X_0 X_{t_n} + \left[\alpha_n X_{t_n}^2 + \beta_n X_{t_n} D_{t_n} + \gamma_n D_{t_n}^2\right]. \tag{15}$$

The associated optimal trading strategy is given by

$$x_n = \frac{1}{2}\delta_{n+1}\left[\epsilon_{n+1} X_{t_n} - \phi_{n+1} D_{t_n}\right] \tag{16}$$

for $n = 0, ..., N-1$ *and* $x_N = X_T$*, where the sequences* α_n*,* β_n *and* γ_n *for* $n = 0, ..., N$ *are given in (11) and* δ_n*,* ϵ_n *and* ϕ_n *for* $n = 1, ..., N$ *can be found in (12).*

Proof: We are considering the control problem (5)

$$C_0 = \min_{\left\{x_0,...,x_N \in \mathbb{R} \mid \sum_{n=0}^N x_n = X_0\right\}} \mathbb{E}\left[\sum_{n=0}^N \overline{P}_{t_n} x_n\right],$$

which we will tackle analogous to the paper of Bertsimas and Lo [5] by using dynamic programming.

In the following we show (15) via (6) and backward induction. The optimal trading strategy (16) follows from this backward induction, too.
In the case of $n = N$, i.e. $t = t_N = T$, we have to buy the remaining X_T shares for a average price of $\overline{P}_T = \left(S_T + \frac{z}{2}\right) + \lambda\left(X_0 - X_T\right) + D_T + \frac{1}{2q}X_T$. Consequently the cost in T is described by

$$\begin{aligned} C_T = \overline{P}_T X_T &= \left(S_T + \frac{z}{2}\right) X_T + \lambda X_0 X_T + \left(\frac{1}{2q} - \lambda\right) X_T^2 + X_T D_T \\ &= \left(S_T + \frac{z}{2}\right) X_T + \lambda X_0 X_T + \left[\alpha_N X_T^2 + \beta_N X_T D_T\right]. \end{aligned}$$

This shows the induction basis.
Let us now consider $t = t_{n-1}$. Then we can make use of dynamic programming, since the optimal control $\{x_0, ..., x_N\}$ has to be optimal for every trading point in time t_n onwards. This thought leads us to the following Bellman equation

$$C_{t_{n-1}} = \min_{x_{n-1} \in \mathbb{R}} \left(\overline{P}_{t_{n-1}} x_{n-1} + \mathbb{E}\left[C_{t_n} | \mathcal{F}_{t_{n-1}}\right]\right), \tag{17}$$

Because of the induction hypothesis and the dynamic given in (7) for D_{t_n} we can form

$$\begin{aligned} C_{t_n} &= \left(S_{t_n} + \frac{z}{2}\right) X_{t_n} + \lambda X_0 X_{t_n} + \left[\alpha_n X_{t_n}^2 + \beta_n X_{t_n} D_{t_n} + \gamma_n D_{t_n}^2\right] \\ &= \left(S_{t_n} + \frac{z}{2}\right)\left(X_{t_{n-1}} - x_{n-1}\right) + \lambda X_0 \left(X_{t_{n-1}} - x_{n-1}\right) + \\ &\quad \Big[\alpha_n \left(X_{t_{n-1}} - x_{n-1}\right)^2 + \beta_n \left(X_{t_{n-1}} - x_{n-1}\right)\left(D_{t_{n-1}} + \kappa x_{n-1}\right) e^{-\rho\tau} + \\ &\quad \gamma_n \left(D_{t_{n-1}} + \kappa x_{n-1}\right)^2 e^{-2\rho\tau}\Big]. \end{aligned}$$

As a reminder, τ stands for the distance between the trading times t_n and t_{n-1}. Plugging this term into (17) and respecting that S is a martingale we find

$$\begin{aligned} C_{t_{n-1}} &= \min_{x_{n-1}\in\mathbb{R}} \Bigg(\left(S_{t_{n-1}} + \frac{z}{2}\right) X_{t_{n-1}} + \lambda X_0 X_{t_{n-1}} - \lambda X_{t_{n-1}} x_{n-1} + D_{t_{n-1}} x_{n-1} \\ &+ \frac{1}{2q} x_{n-1}^2 + \Big[\alpha_n (X_{t_{n-1}} - x_{n-1})^2 + \beta_n \left(X_{t_{n-1}} - x_{n-1}\right)\left(D_{t_{n-1}} + \kappa x_{n-1}\right) e^{-\rho\tau} \\ &+ \gamma_n \left(D_{t_{n-1}} + \kappa x_{n-1}\right)^2 e^{-2\rho\tau}\Big]\Bigg), \end{aligned}$$

where at this stage we used the assumption that X is deterministic.
Let us now define

$$\begin{aligned} f_{n-1}(x_{n-1}) &:= \left(S_{t_{n-1}} + \frac{z}{2}\right) X_{t_{n-1}} + \lambda X_0 X_{t_{n-1}} - \lambda X_{t_{n-1}} x_{n-1} + D_{t_{n-1}} x_{n-1} + \frac{1}{2q} x_{n-1}^2 + \\ &\Big[\alpha_n \left(X_{t_{n-1}} - x_{n-1}\right)^2 + \beta_n \left(X_{t_{n-1}} - x_{n-1}\right)\left(D_{t_{n-1}} + \kappa x_{n-1}\right) e^{-\rho\tau} + \\ &\gamma_n \left(D_{t_{n-1}} + \kappa x_{n-1}\right)^2 e^{-2\rho\tau}\Big]. \end{aligned}$$

Consequently we have $C_{t_{n-1}} = \min_{x_{n-1}\in\mathbb{R}} f_{n-1}(x_{n-1})$. Then f_{n-1} is quadratic in x_{n-1} and can also be written as:

$$\begin{aligned} f_{n-1}(x_{n-1}) &= \delta_n^{-1}\left[x_{n-1} - \frac{1}{2}\delta_n\left(\epsilon_n X_{t_{n-1}} - \phi_n D_{t_{n-1}}\right)\right]^2 + \\ &\left(S_{t_{n-1}} + \frac{z}{2}\right) X_{t_{n-1}} + \lambda X_0 X_{t_{n-1}} + \\ &X_{t_{n-1}}^2\left[\alpha_n - \frac{1}{4}\delta_n\epsilon_n^2\right] + \\ &X_{t_{n-1}} D_{t_{n-1}}\left[\beta_n e^{-\rho\tau} + \frac{1}{2}\delta_n\epsilon_n\phi_n\right] + \\ &D_{t_{n-1}}^2\left[\gamma_n e^{-2\rho\tau} - \frac{1}{4}\delta_n\phi_n^2\right] \end{aligned}$$

We notice that the parabola f_{n-1} has exactly one minimum in

$$x_{n-1} = \frac{1}{2}\delta_n\left(\epsilon_n X_{t_{n-1}} - \phi_n D_{t_{n-1}}\right),$$

since $f''_{n-1}(x_{n-1}) = 2\delta_n^{-1}$ is according to Lemma 3 strictly positive. By plugging this minimum into the parabola f_{n-1}, which is opening to the top, and by having a look at the backward recursions (11) we get the desired term for $C_{t_{n-1}}$. Particularly this proves the existence of a unique optimal trading strategy. □

Remark 5. In order to get a better understanding of the structure of the above dynamic programming principle, we want to highlight the following fact. We have $C_{t_{n-1}} = \min_{x_{n-1}\in\mathbb{R}} f_{n-1}(x_{n-1})$ with

$$\begin{aligned} f_{n-1}(x_{n-1}) &= \left(S_{t_{n-1}} + \frac{z}{2}\right) X_{t_{n-1}} + \lambda X_0 X_{t_{n-1}} + \\ &\alpha_n X_{t_{n-1}}^2 + \beta_n e^{-\rho\tau} X_{t_{n-1}} D_{t_{n-1}} + \gamma_n e^{-2\rho\tau} D_{t_{n-1}}^2 + \\ &\delta_n^{-1} x_{n-1}^2 - \epsilon_n x_{n-1} X_{t_{n-1}} + \phi_n x_{n-1} D_{t_{n-1}} \end{aligned}$$

That is α, β and γ are the sequences belonging to X^2, XD and D^2 and δ^{-1}, ϵ and ϕ belong to x^2, xX and xD. This systematic will show up with slightly different sequences $\alpha, ..., \phi$ in the following backward inductions involving dynamic programming, too.

So far we have not checked if the optimal strategy given in (16) has positive x_n for $n = 0, ..., N$. But this has to be met, since we neglected the left hand side of the LOB as mentioned in Section 2, i.e. sales of shares are not allowed in our model. Therefore, we work out the optimal strategy (16) explicitly in Lemma 7 in order to directly derive the desired non-negativity. Again we need an auxiliary lemma to do so:

Lemma 6. *(The temporary price impact is constant under the optimal strategy)* *Let us assume we choose the strategy* $x_0 = X_0 \frac{1}{(N-1)(1-e^{-\rho\tau})+2}$ *and* $x_n = \frac{X_0 - 2x_0}{N-1}$ *for* $n = 1, ..., i-1$. *Then*

$$D_{t_i} = \kappa x_0 e^{-\rho\tau}$$

Proof: We show the lemma by forward induction over i and use that the process D has the dynamic $D_0 = 0$ and $D_{t_n} = \left(D_{t_{n-1}} + \kappa x_{n-1}\right) e^{-\rho\tau}$ for $n = 1, ..., N$ as given in (7).
We get immediately $D_{t_1} = \kappa x_0 e^{-\rho\tau}$ and the following induction step, where we know according to the induction hypothesis that $D_{t_{i-1}} = \kappa x_0 e^{-\rho\tau}$:

$$D_{t_i} = \left(D_{t_{i-1}} + \kappa x_{i-1}\right) e^{-\rho\tau} = \left(\kappa x_0 e^{-\rho\tau} + \kappa \frac{X_0 - 2x_0}{N-1}\right) e^{-\rho\tau} = \kappa x_0 e^{-\rho\tau}$$

We get the last equation by plugging in x_0 explicitly. $\square$

Lemma 7. *(Non-negativity and explicit form of the optimal strategy)* *The optimal strategy to be found in (16) of Lemma 4 can be written explicitly as*

$$\begin{aligned} x_0 &= \quad x_N \quad = X_0 \frac{1}{(N-1)\left(1-e^{-\rho\tau}\right)+2} \quad \textit{and} \\ x_n &= \frac{X_0 - 2x_0}{N-1} = X_0 \frac{1-e^{-\rho\tau}}{(N-1)\left(1-e^{-\rho\tau}\right)+2} \quad \textit{for } n = 1, ..., N-1. \end{aligned}$$

Proof: According to Lemma 4 the first trade of the optimal strategy is

$$x_0 = \frac{1}{2}\delta_1[\epsilon_1 X_0 - \phi_1 D_0] = \frac{1}{2} X_0 \delta_1 \epsilon_1$$

because of $D_0 = 0$. By inserting the explicit values of δ_1 and ϵ_1 from Lemma 2, we get as desired by direct calculation

$$x_0 = X_0 \frac{e^{2\rho\tau}\left(e^{\rho\tau} - e^{-\rho\tau}\right)}{\left[(N-1)\left(1-e^{2\rho\tau}\right) + (N+1)\left(e^{3\rho\tau} - e^{\rho\tau}\right)\right]} = X_0 \frac{1}{(N-1)\left(1-e^{-\rho\tau}\right)+2}.$$

Analogously, we can now consider x_1 bearing in mind that $X_{t_1} = X_0 - x_0$ and $D_{t_1} = \kappa x_0 e^{-\rho\tau}$:

$$\begin{aligned} x_1 &= \frac{1}{2}\delta_2\left[\epsilon_2 X_{t_1} - \phi_2 D_{t_1}\right] \\ &= \frac{\left(e^{3\rho\tau} - e^{\rho\tau}\right)\left(X_0 - x_0\right) - x_0\left[(N-1)\left(e^{2\rho\tau} - 1\right) - (N-2)\left(e^{\rho\tau} - e^{-\rho\tau}\right)\right]}{(N-2) - Ne^{\rho\tau} - (N-2)e^{2\rho\tau} + Ne^{3\rho\tau}} \\ &= X_0 \frac{1-e^{-\rho\tau}}{(N-1)\left(1-e^{-\rho\tau}\right)+2}, \end{aligned}$$

where again a lengthy arithmetic calculation is involved in the last step.

We can once more do a forward induction in order to show

$$x_n = \frac{X_0 - 2x_0}{N-1} \quad \text{for} \quad n = 1, ..., N-1.$$

Since we have already considered x_1, the induction basis is already proved. According to the induction hypothesis we know that $x_1 = ... = x_{i-1} = \frac{X_0 - 2x_0}{N-1}$ and we can therefore use Lemma 6, which states that $D_{t_i} = \kappa x_0 e^{-\rho\tau}$. Hence we get the following induction step by using the explicit formulas for δ, ϵ and ϕ out of Lemma 2 and by plugging in $x_0 = X_0 \frac{1}{(N-1)(1-e^{-\rho\tau})+2}$:

$$\begin{aligned}
x_i &= \frac{1}{2}\delta_{i+1}\left[\epsilon_{i+1}\left(X_0 - x_0 - (i-1)\frac{X_0 - 2x_0}{N-1}\right) - \phi_{i+1}\kappa x_0 e^{-\rho\tau}\right] \\
&= \frac{X_0\left(e^{3\rho\tau} - e^{\rho\tau}\right)\frac{N-i}{N-1}}{(N-i-1)\left(1-e^{2\rho\tau}\right) + (N-i+1)\left(e^{3\rho\tau} - e^{\rho\tau}\right)} \\
&+ \frac{x_0\left[\left(e^{3\rho\tau} - e^{\rho\tau}\right)\frac{2i-N-1}{N-1} - (N-i)\left(e^{2\rho\tau} - 1\right) + (N-i-1)\left(e^{\rho\tau} - e^{-\rho\tau}\right)\right]}{(N-i-1)\left(1-e^{2\rho\tau}\right) + (N-i+1)\left(e^{3\rho\tau} - e^{\rho\tau}\right)} \\
&= X_0\frac{1-e^{-\rho\tau}}{(N-1)\left(1-e^{-\rho\tau}\right)+2}
\end{aligned}$$

In the end we have of course

$$x_N = X_0 - x_0 - (N-1)\frac{X_0 - 2x_0}{N-1} = x_0.$$

□

This ultimately proves Proposition 1.

3.3 Alternative models

In order to get a better insight into the Obizhaeva and Wang model [20], which we frequently use throughout this thesis, this subsection describes an alternative model where the whole impact is linear and permanent. At the same time this digression serves as a motivation for Subsection 3.4 where the Obizhaeva and Wang model is extended by a second component of temporary impact.

As mentioned at the beginning of Chapter 3, some authors like e.g. Almgren and Chriss in [2] work with fixed discrete trading times, instead of optimising over them as well by taking continuous trading time. In the simplest case introduced by Bertsimas and Lo [5], the whole price impact is modelled linearly with constant λ. Accordingly, the average price per share at time t_n is

$$\overline{P}_{t_n} = \left(S_{t_n} + \frac{z}{2}\right) + \lambda\sum_{i=0}^{n} x_i. \tag{18}$$

Comparing this with the average price in the LOB model of Obizhaeva and Wang given by (6) and (7), (18) is a special case of the LOB model with

$$\lambda = \kappa = \frac{1}{2q} \quad \text{and} \quad \rho = \infty.$$

When we now consider the optimising problem (5) in this special case, we obtain

$$C_0 = \left(S_0 + \frac{z}{2}\right) X_0 + \min_{\left\{x_0,\dots,x_N \in \mathbb{R} \mid \sum_{n=0}^N x_n = X_0\right\}} \lambda \sum_{n=0}^{N} \left(x_n \sum_{i=0}^{n} x_i \right). \tag{19}$$

Rearranging the sum and setting $X_{t_{N+1}} := 0$, a short calculation yields

$$\begin{aligned} &\sum_{n=0}^{N} \left(x_n \sum_{i=0}^{n} x_i \right) = \sum_{n=0}^{N} x_n X_{t_n} = \sum_{n=0}^{N} \left(X_{t_n} - X_{t_{n+1}} \right) X_{t_n} = \\ &\frac{1}{2} \sum_{n=0}^{N} \left[X_{t_n}^2 - X_{t_{n+1}}^2 + \left(X_{t_n} - X_{t_{n+1}} \right)^2 \right] = \frac{1}{2} X_0^2 + \frac{1}{2} \sum_{n=0}^{N} x_n^2. \end{aligned} \tag{20}$$

Now it is obvious that the term x_n occurs quadratic in C_0. Therefore, the minimum takes place in $x_0 = \dots = x_N = \frac{X_0}{N+1}$. This means that in this special case it is optimal to spread the purchase of the X_0 shares evenly over the $N+1$ trading times. Contrary to Figure 5 there are no large trades in $t = 0$ and T. We obtain

$$C_0 = \left(S_0 + \frac{z}{2}\right) X_0 + \frac{\lambda}{2} X_0^2 \left(1 + \frac{1}{N+1}\right) = \left(S_0 + \frac{z}{2}\right) X_0 + \frac{X_0^2}{4q} \frac{N+2}{N+1}.$$

If we try now to optimise over the trading times in (19) as well by letting τ converge to zero or N to infinity respectively, this will lead to the following problem: If we take an arbitrary continuously differentiable trading strategy $(X_t)_{t\in[0,T]}$ with $X_T = 0$ and set $x_n = X_{t_n} - X_{t_{n+1}}$ for $n = 0, \dots, N$, then all these strategies will cause the same cost

$$\left(S_0 + \frac{z}{2}\right) X_0 + \frac{\lambda}{2} X_0^2.$$

This is due to the fact that the term from (20) converges to zero

$$\lim_{N\to\infty} \sum_{n=0}^{N} x_n^2 = \lim_{N\to\infty} \sum_{n=0}^{N} \tau^2 \left(\frac{X_{t_{n+1}} - X_{t_n}}{\tau} \right)^2 = 0,$$

since the derivative of X is bounded on $[0, T]$ and $\tau = \frac{T}{N}$. This is definitely not realistic. E.g. one could shift excessively much trading forward in time without any cost increase as long as the strategy stays continuous. Thus, having only permanent impact is not satisfying.

In order to counteract this problem, Almgren and Chriss [2] as well as Huberman and Stanzl [14] introduce a temporary price impact $h(\frac{x_n}{\tau})$ in order to penalize continuous trading with high intensities. At the same time two components of price impact are empirically more reasonable. That is

$$\overline{P}_{t_n} = \left(S_{t_n} + \frac{z}{2}\right) + \lambda \sum_{i=0}^{n} x_i + h(\frac{x_i}{\tau})$$

with an increasing function h, where compared to the model of Obizhaeva and Wang [20] the temporary impact decays immediately instead of exponentially. The function h is modelled linearly in the simplest case: $h(\frac{x_n}{\tau}) = \frac{\eta}{\tau}x_n$ with a positive constant η. In [3], Almgren and Chriss presume the form $h(v) = \eta v^\beta$ and determine $\beta = \frac{3}{5}$ empirically. Nevertheless, there remain two disadvantages when the temporary impact is only modelled by the function h: In contrast to the temporary impact in the LOB model, $h(\frac{x_n}{\tau})$ is not influencing the succeeding prices. This is especially unsatisfying when the distance between trading times τ is small. Furthermore, discrete trading is restrained because the temporary impact $h(\frac{x_n}{\tau})$ for a discrete trade $x_n > 0$ at t_n would converge to infinity for $\tau \to 0$.

However, the question arises how the optimal strategy will behave if we combine the two different types of temporary impacts mentioned above. We address this topic in the next subsection.

3.4 Two components of temporary impact

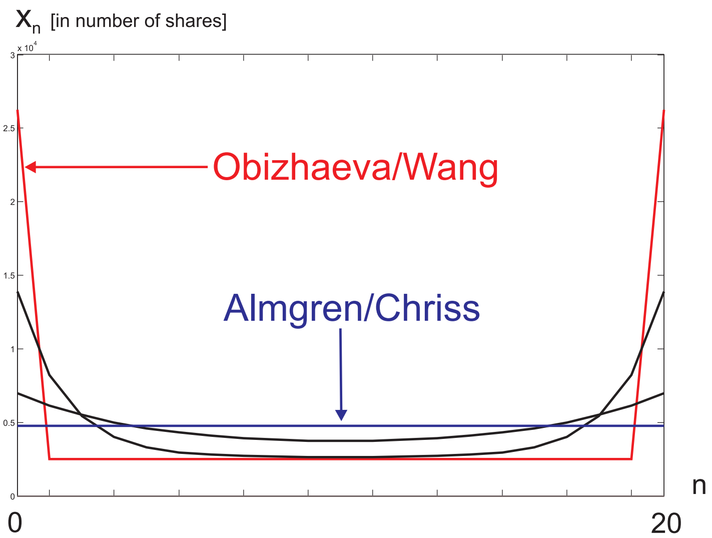


Figure 8: Optimal strategies $x_0, ..., x_N$ for $\eta = 0$, 10^{-6}, 10^{-5} and 1, respectively. The Obizhaeva and Wang model corresponds to $\eta = 0$ and for large η we get the Almgren and Chriss result. The other parameters are set to $T = 1$, $N = 20$, $X_0 = 100{,}000$, $\rho = 2$ and $\lambda = \kappa = 1/10{,}000$.

We now consider a model with two components of linear temporary impact, which represents a mixture of the models of Obizhaeva and Wang [20] as well as Almgren and Chriss [2]. In this combined model, the average price incorporates not only the permanent impact and an exponentially decaying linear temporary impact as in the

LOB model of Obizhaeva and Wang, but also an only instantaneously existing linear temporary impact $h(\frac{x_n}{\tau}) = \frac{\eta}{\tau}x_n$. Therefore it can be written as

$$\begin{aligned}\overline{P}_{t_n} &= \left(S_{t_n} + \frac{z}{2}\right) + \lambda\left(X_0 - X_{t_n}\right) + D_{t_n} + \frac{1}{2q}x_n + \frac{\eta}{\tau}x_n \\ &= \left(S_{t_n} + \frac{z}{2}\right) + \lambda\left(X_0 - X_{t_n}\right) + D_{t_n} + \widetilde{\eta}x_n,\end{aligned}$$

where we used (6) and defined

$$\widetilde{\eta} := \frac{\lambda + \kappa}{2} + \frac{\eta}{\tau}.$$

The processes S, X and D are defined as before. In comparison to (6), the term $\frac{1}{2q}x_n$ is replaced by $\widetilde{\eta}x_n$. We can calculate the optimal strategy by using Lemma 4 and replacing $\frac{1}{2q}$ by $\widetilde{\eta}$ in α_N and δ_n. The optimal strategies $x_0, ..., x_N$ for different choices of η are given in Figure 8. When we set $\eta = 0$, we are in the Obizhaeva and Wang framework and our optimal strategy consists of two discrete trades x_0, x_N and constant trading in between. When we select high values of η, we get $x_0 \approx ... \approx x_N$, which is the simple Almgren and Chriss case as derived in the last section. For moderate values of η, we get a U-shaped pattern for the optimal strategy $x_0, .., x_N$. These U-shaped strategies are marked in black in Figure 8. All strategies are symmetric in time, in particular $x_0 = x_N$.
Figure 9 shows that the optimal strategy remains U-shaped for $\tau \to 0$ or $N \to \infty$ respectively.

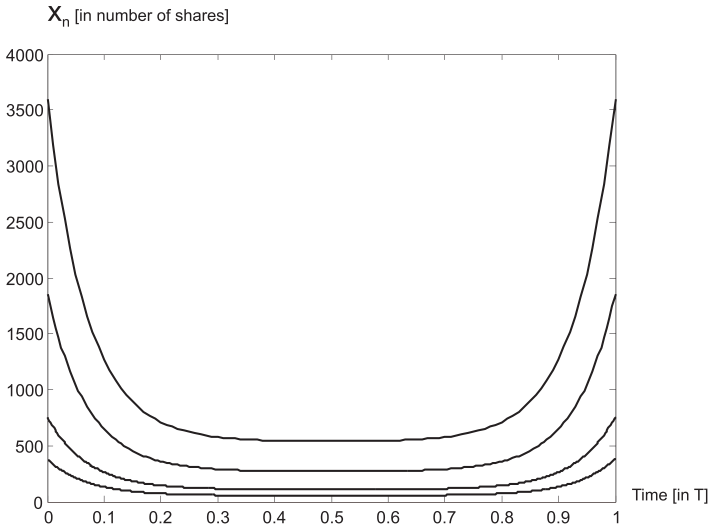


Figure 9: Optimal strategies $x_0, ..., x_N$ for $\eta = 10^{-6}$ and $N = 100, 200, 500, 1000$ top down. The other parameters are chosen as in Figure 8. On the x-axis we plotted the time instead of n, since N varies.

3.5 Variance of the cost

We take again the Obizhaeva and Wang pricing model and assume that the buyer is risk averse. This means that he not only wants to minimise the expectation of the cost resulting from the purchase of the X_0 shares, but also the risk of deviations from this expectation. We choose the variance of the cost to quantify this risk and consider the following mean-variance optimising problem instead of (5)

$$C_0 = \min_{\{x_0,\dots,x_N \in \mathbb{R} \mid \sum_{n=0}^{N} x_n = X_0\}} \left\{ \mathbb{E}\left[\sum_{n=0}^{N} \overline{P}_{t_n} x_n\right] + \frac{1}{2}\, a \operatorname{Var}\left(\sum_{n=0}^{N} \overline{P}_{t_n} x_n\right)\right\}. \tag{21}$$

With the constant $a \geq 0$ we denominate the risk aversion coefficient of the institutional investor and we take $\overline{P}_{t_n}$ as given in (6).
We show in the following lemma how the variance in (21) can be simplified.

Lemma 8 (Variance in discrete time)**.** *The cost of the trading strategies $x_0, \dots, x_N \in \mathbb{R}$ with $x_N = X_T$ feature the following variance:*

$$\operatorname{Var}(\sum_{n=0}^{N} \overline{P}_{t_n} x_n) = \sigma^2 \tau \sum_{n=1}^{N} X_{t_n}^2$$

Proof: By taking into account that S is the only stochastic process in $\overline{P}_{t_n}$, we can readily evaluate

$$\operatorname{Var}\left(\sum_{n=0}^{N} \overline{P}_{t_n} x_n\right) = \operatorname{Var}\left(\sum_{n=0}^{N} S_{t_n} x_n\right) = \operatorname{Var}\left(\sum_{n=0}^{N} S_{t_n}\left(X_{t_n} - X_{t_{n+1}}\right)\right) =$$
$$\operatorname{Var}\left(S_0 X_0 + \sum_{n=1}^{N}\left(S_{t_n} - S_{t_{n-1}}\right) X_{t_n}\right) = \sum_{n=1}^{N} \operatorname{Var}\left(\sigma\left(W_{t_n} - W_{t_{n-1}}\right) X_{t_n}\right) = \sum_{n=1}^{N} \sigma^2 X_{t_n}^2 \tau.$$

$\square$

We get the following generalisation of Lemma 4.

Corollary 9. *(Optimal trading strategy in discrete time with risk aversion)*
The combination of expectation and variance of the cost under the optimal strategy as given in (21) is

$$C_{t_n} = \left(S_{t_n} + \frac{z}{2}\right) X_{t_n} + \lambda X_0 X_{t_n} + \left[\widetilde{\alpha}_n X_{t_n}^2 + \widetilde{\beta}_n X_{t_n} D_{t_n} + \widetilde{\gamma}_n D_{t_n}^2\right]. \tag{22}$$

The associated optimal strategy is given by

$$x_n = \frac{1}{2}\widetilde{\delta}_{n+1}\left[\widetilde{\epsilon}_{n+1} X_{t_n} - \widetilde{\phi}_{n+1} D_{t_n}\right] \tag{23}$$

for $n = 0, \dots, N-1$ and $x_N = X_T$. The parameters in (22) and (23) are given recursively by

$$\begin{aligned}
\widetilde{\alpha}_N = \frac{1}{2q} - \lambda \quad \textit{and} \quad \widetilde{\alpha}_n &= \widetilde{\alpha}_{n+1} - \frac{1}{4}\widetilde{\delta}_{n+1}\widetilde{\epsilon}_{n+1}^2 + \frac{1}{2} a\sigma^2\tau \\
\widetilde{\beta}_N = 1 \quad \textit{and} \quad \widetilde{\beta}_n &= \widetilde{\beta}_{n+1} e^{-\rho\tau} + \frac{1}{2}\widetilde{\delta}_{n+1}\widetilde{\epsilon}_{n+1}\widetilde{\phi}_{n+1} \\
\widetilde{\gamma}_N = 0 \quad \textit{and} \quad \widetilde{\gamma}_n &= \widetilde{\gamma}_{n+1} e^{-2\rho\tau} - \frac{1}{4}\widetilde{\delta}_{n+1}\widetilde{\phi}_{n+1}^2
\end{aligned}$$

for $n = 0, ..., N$ *and*

$$\begin{aligned}
\widetilde{\delta}_n &= \left(\frac{1}{2q} + \widetilde{\alpha}_n - \widetilde{\beta}_n \kappa e^{-\rho\tau} + \widetilde{\gamma}_n \kappa^2 e^{-2\rho\tau} + \frac{1}{2} a\sigma^2\tau\right)^{-1} \\
\widetilde{\epsilon}_n &= \lambda + 2\widetilde{\alpha}_n - \widetilde{\beta}_n \kappa e^{-\rho\tau} + a\sigma^2\tau \\
\widetilde{\phi}_n &= 1 - \widetilde{\beta}_n e^{-\rho\tau} + 2\widetilde{\gamma}_n \kappa e^{-2\rho\tau}
\end{aligned}$$

for $n = 1, ..., N$.

Proof: The proof of Corollary 9 can be done analogously to the proof of Lemma 4. But in the induction step we get because of Lemma 8

$$C_{t_{n-1}} = \min_{x_{n-1} \in \mathbb{R}} \left(\overline{P}_{t_{n-1}} x_{n-1} + \frac{1}{2} a\sigma^2 \tau X_{t_n}^2 + \mathbb{E}\left[C_{t_n} | \mathcal{F}_{t_{n-1}}\right] \right).$$

Because of

$$X_{t_n}^2 = \left(X_{t_{n-1}} - x_{n-1}\right)^2 = X_{t_{n-1}}^2 - 2x_{n-1} X_{t_{n-1}} + x_{n-1}^2$$

and Remark 5 there are only slightly changes in comparison to Lemma 4. We only have to modify α, δ and ϵ as given in Corollary 9. □

In comparison to the sequences in the proof of Proposition 1, we only have to incorporate the extra term $\frac{1}{2}a\sigma^2\tau$ in case of α and δ, and the term $a\sigma^2\tau$ in case of ϵ.

Remark 10. In the following chapters, which are dealing with auctions, we often make use of the dynamic programming principle in order to optimise the expectation of the cost. Although we are not including a supplementary consideration of the variance analogously to Corollary 9 there, it should be possible to do so.

The optimal strategy according to Corollary 9 is illustrated in Figure 10. It does not only depend on the parameters ρ and T, but also on κ, a and σ and is consequently more complex than the strategy from Proposition 1. The figure shows that the increase of the risk aversion a or the choice of a higher volatility σ leads to a shift of the trading forward in time, which is exactly what one would expect.
So far we have concentrated ourselves on discrete trading time. We derived and explained the trading strategy as given in Figure 5, which minimises the expected cost. Furthermore, we combined our model with the one of Almgren and Chriss and incorporated the variance of the cost by a mean-variance ansatz.

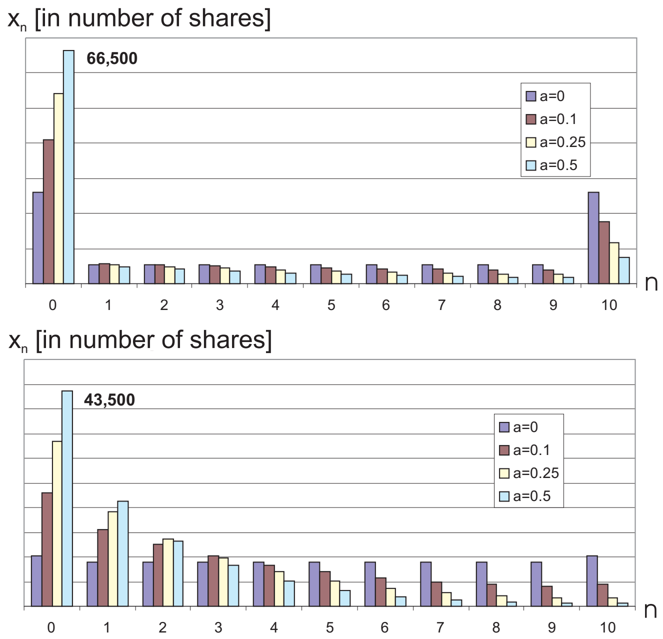


Figure 10: Optimal trading strategies with risk aversion in discrete time for $N = 10$. Analogous to the first diagram in Figure 5, x_0 to x_{10} are plotted, but for different risk aversion coefficients a. In the upper picture we set $\rho = 2.31$ and $\rho = 20$ in the lower one. We fixed $\sigma = 0.025$ and the remaining parameters can be found in the appendix.

4 Optimal trading strategy in continuous time

Here we will again discuss the problem described in the previous chapter, but we will allow for trading in continuous time.

Definition 11. (Trading strategy X) The deterministic, left-continuous and decreasing process $(X_t)_{t\in[0,T]}$ of shares still to be bought is called **trading strategy**.

We want to consider trading strategies that can be described by their density and their jumps, which will become clear in the following definition.

Definition 12. (Buying intensity and jumps of X)
Let $x_t := X_t - X_{t+}$ be the jump of the trading strategy X at time t and

$$\mathcal{T} := \{t \in [0,T] | x_t \neq 0\}$$

the set of the **jumping times**. We call the positive function μ with

$$X_0 - X_t = \int_0^t \mu_u du + \sum_{\substack{u\in\mathcal{T}\\ u<t}} x_u$$

buying intensity or continuous part of the trading strategy.
Let Θ be the set of all tuples $\{(\mu_t)_{t\in[0,T]}, (x_t)_{t\in\mathcal{T}}\}$.

The process X is left-continuous and decreasing on the compact interval $[0,T]$. Therefore, $\mathcal{T}$ is countable and the sum in the definition is well defined. We know that each trading strategy X can be identified with a tuple $\{(\mu_t)_{t\in[0,T]}, (x_t)_{t\in\mathcal{T}}\}$ and vice versa. Therefore, we will use these two expressions synonymously.

Using the terms introduced in Definition 12, we can formulate our best ask price in the LOB model at time t as

$$A_t^\Theta = \left(S_t + \frac{z}{2}\right) + \lambda\left(X_0 - X_t\right) + D_t. \tag{24}$$

The superscript Θ is meant to reveal the fact that the process depends on the considered trading strategy X. Similar to the discrete time case, the best ask price in (24) is made up of four components:
First of all we have the equilibrium price S_t, which does not depend on the considered strategy. The second component is half of the spread z. The permanent price impact, which is linear in the number of shares already bought until t, is the third component. Finally we have the temporary impact D_t or more precisely D_t^Θ. Analogously to the discrete time case (7) it is given by

$$D_0 = 0 \quad \text{and} \quad D_t = \int_0^t \left(-\rho D_u + \kappa\mu_u\right) du + \kappa \sum_{u\in\mathcal{T}, u<t} x_u. \tag{25}$$

The total cost for a fixed strategy X is then

$$\widetilde{C}_0^\Theta := \int_0^T A_u^\Theta \mu_u du + \sum_{u\in\mathcal{T}} \left(A_u^\Theta + \frac{1}{2q} x_u\right) x_u$$

and we look for the trading strategy that causes minimal expected cost such that we get the following optimising problem

$$C_0 := \min_\Theta \mathbb{E}\left[\widetilde{C}_0^\Theta\right]. \tag{26}$$

In the proposition below, which is also found in the Obizhaeva and Wang paper [20], we state the optimal trading strategy with the associated cost process C_t. It is a special case of Proposition 15, which we will prove later on using control theory. Nevertheless, we want to give an intuition in Remark 14 how to derive Proposition 13 from the discrete time case as stated in Proposition 1 by letting τ converge to zero. This is only possible thanks to having explicit forms of the sequences α to ϕ.

Proposition 13. *(Optimal trading strategy in continuous time)*
The expected cost under the optimal strategy is

$$C_t = \left(S_t + \frac{z}{2}\right) X_t + \lambda X_0 X_t + \left[\alpha_t X_t^2 + \beta_t X_t D_t + \gamma_t D_t^2\right] \quad \textit{with}$$

$$\alpha_t = \frac{\kappa}{\rho(T-t)+2} - \frac{\lambda}{2}, \quad \beta_t = \frac{2}{\rho(T-t)+2} \quad \textit{and} \quad \gamma_t = \frac{-\rho(T-t)}{2\kappa\left[\rho(T-t)+2\right]}.$$

The associated optimal trading strategy is given by

$$x_0 = x_T = \frac{X_0}{\rho T + 2} \quad \textit{and} \quad \mu_t \equiv \frac{\rho X_0}{\rho T + 2} \quad \textit{for } t \in (0,T).$$

Remark 14. Proposition 13 can be deduced from Proposition 1 when we let the step size τ converge to zero or the number of steps $N = \frac{T}{\tau}$ to infinity. In this way we get the optimal strategy in continuous trading time by using L'Hospital's Rule:

$$\lim_{N\to\infty} x_0 = \lim_{N\to\infty} \frac{X_0}{(N-1)\left(1-e^{-\rho\frac{T}{N}}\right)+2} = \frac{X_0}{\rho T + 2}$$

and for $n = 1, ..., N-1$ we have

$$\lim_{N\to\infty} x_n = \lim_{N\to\infty} X_0 \frac{1-e^{-\rho\frac{T}{N}}}{(N-1)\left(1-e^{-\rho\frac{T}{N}}\right)+2} = 0.$$

Therefore, the intensity of trading is

$$\mu_t = \frac{X_0 - 2\frac{X_0}{\rho T+2}}{T} = \frac{\rho X_0}{\rho T + 2}.$$

Let us now examine the terms α, β and γ. Setting $N = \frac{T}{\tau}$ and $n = \frac{t}{\tau}$ we get

$$\lim_{\tau\to 0}(N-n)\left(e^{\rho\tau}-1\right) = (T-t)\lim_{\tau\to 0}\frac{e^{\rho\tau}-1}{\tau} = \rho(T-t).$$

Therefore, the limiting behaviour as $\tau \to 0$ can be easily identified for α, β and γ as

$$\begin{aligned}
\lim_{\tau\to 0}\alpha_n &= \lim_{\tau\to 0}\frac{\frac{1}{2q}\left(1+e^{\rho\tau}\right) - \frac{\lambda}{2}\left[(N-n)\left(e^{\rho\tau}-1\right)+2\left(1+e^{\rho\tau}\right)\right]}{\left[(N-n)\left(e^{\rho\tau}-1\right)+\left(1+e^{\rho\tau}\right)\right]} \\
&= \frac{\frac{1}{q} - \frac{\lambda}{2}\left[\rho(T-t)+4\right]}{\rho(T-t)+2} = \frac{\kappa}{\rho(T-t)+2} - \frac{\lambda}{2} = \alpha_t \\
\lim_{\tau\to 0}\beta_n &= \lim_{\tau\to 0}\frac{1+e^{\rho\tau}}{\left[(N-n)\left(e^{\rho\tau}-1\right)+\left(1+e^{\rho\tau}\right)\right]} = \frac{2}{\rho(T-t)+2} = \beta_t \\
\lim_{\tau\to 0}\gamma_n &= \frac{1}{2\kappa}\lim_{\tau\to 0}\frac{-(N-n)\left(e^{\rho\tau}-1\right)}{\left[(N-n)\left(e^{\rho\tau}-1\right)+\left(1+e^{\rho\tau}\right)\right]} = \frac{-\rho(T-t)}{2\kappa\left[\rho(T-t)+2\right]} = \gamma_t.
\end{aligned}$$

If we again want to incorporate the risk aversion of the institutional investor in form of a risk aversion coefficient $a \in \mathbb{R}_{\geq 0}$ into the objective function to be minimised, we will obtain instead of (26)

$$C_0 = \min_{\Theta}\left\{\mathbb{E}\left[\widetilde{C}_0^{\Theta}\right] + \frac{1}{2}a\ \operatorname{Var}\left(\widetilde{C}_0^{\Theta}\right)\right\}. \tag{27}$$

Proposition 15. *(Optimal trading strategy in continuous time with risk aversion) The cost process under the optimal strategy is*

$$C_t = \left(S_t + \frac{z}{2}\right)X_t + \lambda X_0 X_t + \left[\alpha_t X_t^2 + \beta_t X_t D_t + \gamma_t D_t^2\right].$$

ρ	Half-life $\frac{\log 2}{\rho}$	$x_0 = x_T$ [in number of shares]	Trading in $(0,T)$ [in number of shares]	Continuous trading
0.001	693 days	49,975	50	0.05%
0.5	1.39 days	40,000	20,000	20%
1	270 min	33,333	33,334	33%
2	135 min	25,000	50,000	50%
5	54 min	14,286	71,428	71%
20	13.5 min	4,545	90,910	91%
50	5.4 min	1,923	96,154	96%
1000	0.3 min	100	99,800	99.8%

Table 1: Optimal trading strategy in continuous time for different ρ and half-lives ϑ respectively ($e^{-\rho\vartheta} = \frac{1}{2}$). The risk aversion coefficient is $a = 0$. The parameter choice is given in Appendix A.1.

The associated optimal trading strategy is given by

$$\begin{aligned} x_0 &= X_0 \frac{\kappa f'(0) + a\sigma^2}{\kappa\rho f(0) + a\sigma^2} \quad \textit{and} \quad x_T = X_T = X_0 - x_0 - \int_0^T \mu_u du, \\ \mu_t &= x_0 \kappa \frac{\rho g(t) - g'(t)}{1 + \kappa g(t)} \exp\Big(-\int_0^t \frac{\kappa g'(u) + \rho}{1 + \kappa g(u)} du\Big) \; \textit{for } t \in (0,T). \end{aligned}$$

The following coefficients and functions were used:

$$\begin{aligned} \alpha_t &= \frac{\kappa f(t) - \lambda}{2}, \quad \beta_t = f(t), \quad \gamma_t = \frac{f(t) - 1}{2\kappa}, \\ f(t) &= \frac{v - a\sigma^2}{\kappa\rho} + \left[\Big(\frac{\kappa\rho}{2v} - \frac{\kappa\rho}{v - a\sigma^2 - \kappa\rho}\Big) \exp\Big(\frac{2\rho v(T-t)}{2\kappa\rho + a\sigma^2}\Big) - \frac{\kappa\rho}{2v} \right]^{-1}, \\ v &= \sqrt{a^2\sigma^4 + 2a\sigma^2\kappa\rho}, \\ g(t) &= \frac{\rho f(t) - f'(t)}{\kappa f'(t) + a\sigma^2}. \end{aligned}$$

According to this, Proposition 13 is a special case ($a = 0$) of the proposition above. The proof of it can be found in Section 4.2, which is a revised version of the one to be found in [20]. We first want to explain the results given in Proposition 13 and 15.

4.1 Interpretation of Proposition 13 and 15

Analogously to the discrete time case, the optimal strategy given in Proposition 13 comprises two equally sized discrete trades at 0 and T and a constant allocation of the remaining shares on the time interval $(0,T)$. The buying intensity is constant.
It is notable that the optimal trading strategy from Proposition 13 only depends on the parameters X_0, ρ and T. Since the permanent impact is modelled linearly with constant λ, its influence is the same for all strategies. Therefore, λ is indeed relevant in order to determine the optimal cost, but it does not affect the optimal strategy.

According to Proposition 13, the proportion of the continuous trading in comparison to the total trade is

$$\frac{\int_0^T \mu_t dt}{X_0} = \frac{\rho T}{\rho T + 2}.$$

For the unrealistic case $\rho \to 0$ we only have the two discrete trades at the boundaries of the considered time interval $[0, T]$ and for $\rho \to \infty$ there is only continuous trading. This observation is illustrated in Table 1. The bigger ρ or T are chosen, the smaller are x_0, x_T and the expected cost C_0. This is intuitively clear: Due to a higher resiliency ρ, x_0 can be reduced and the stronger flow of new limit orders can be used by the continuous trading to an increasing degree. For large T we can likewise profit from the longer flow of new limit orders by increasing the continuous trading.

In the case where the investor has a risk aversion $a \neq 0$, we get from Proposition 15 that the larger a or σ are, the more trading should be shifted forward in time. Hence, $x_0 > x_T$ and the buying intensity μ_t is not constant anymore but rather it decreases. This is illustrated in Figure 11 which is in line with Figure 10 for big N.

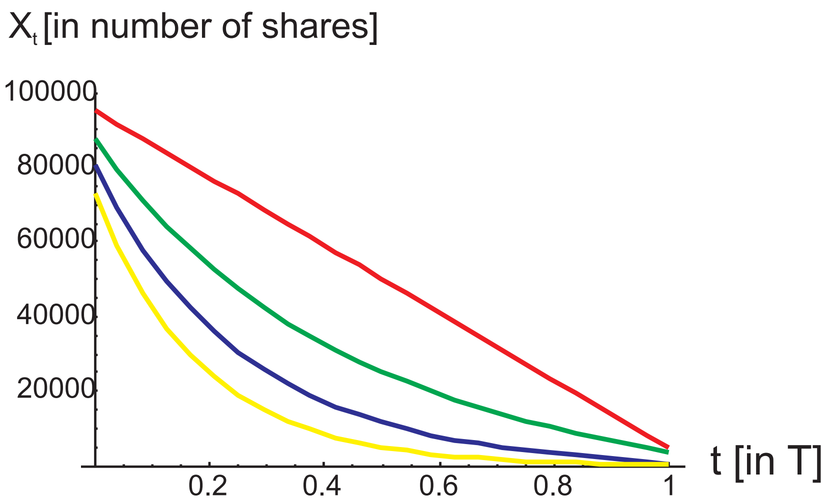


Figure 11: Optimal trading strategies for different risk aversion coefficients $a = 0$, $a = 0.1$, $a = 0.25$ and $a = 0.5$ (from top to bottom). The volatility has been set to $\sigma = 0.025$ and the resiliency parameter is $\rho = 20$. The remaining parameters are selected as in the list in Appendix A.1.

4.2 Proof of Proposition 15 and particularly Proposition 13

Let us now turn to the proof of Proposition 15. As a short recap, we are interested in the best ask price explained in (24) or specifically in the term

$$\widetilde{C}_t^{\Theta} = \int_t^T A_u^{\Theta} \mu_u du + \sum_{u \in \mathcal{T}, u \geq t} \left[A_u^{\Theta} + \frac{1}{2q} x_u \right] x_u,$$

in order to solve the optimisation problem

$$C_t = \min_{\Theta} \left\{ \mathbb{E}\left[\widetilde{C}_t^{\Theta} | \mathcal{F}_t \right] + \frac{1}{2} a \;\operatorname{Var}\left(\widetilde{C}_t^{\Theta} | \mathcal{F}_t \right) \right\}. \tag{28}$$

Thereby the following lemma holds for the variance of $\widetilde{C}_t^{\Theta}$, which we will use later on.

Lemma 16. *(Variance in continuous time) The deterministic, left-continuous, decreasing and positive strategies $(X_u)_{u\in[t,T]}$ with $x_T = X_T$ that are considered by us, feature the following variance:*

$$\operatorname{Var}\left(\widetilde{C}_t^{\Theta}|\mathcal{F}_t\right) = \int_t^T \sigma^2 X_u^2 du.$$

Proof: In Lemma 8 we have already proved the claim in discrete time. Since we only consider deterministic strategies X that are decreasing on the closed interval $[0,T]$, the set of the points of discontinuity of X is a null set. Thus, X is Riemann-integrable and $\tau \to 0$ gives the desired term $\int_0^T \sigma^2 X_t^2 dt$.

An alternative way to prove the lemma is to calculate the variance directly by applying the multidimensional Itô formula for jump processes (Oksendal and Sulem [21], Theorem 1.16) or more exactly by using integration by parts to compute

$$S_T X_T - S_0 X_0 = \int_0^T X_u dS_u - \int_0^T S_u \mu_u du - \sum_{\substack{u\in\mathcal{T}\\ u<T}} S_u x_u.$$

Therefore, we may write

$$\begin{aligned}\operatorname{Var}\left(\widetilde{C}_0^{\Theta}\right) &= \operatorname{Var}\left(\int_0^T S_u\mu_u du + \sum_{u\in\mathcal{T}} S_u x_u\right) = \operatorname{Var}\left(\int_0^T X_u dS_u\right) \qquad (29)\\ &= \sigma^2 \operatorname{Var}\left(\int_0^T X_u dW_u\right) = \sigma^2 \int_0^T X_u^2 du.\end{aligned}$$

Thereby $dS_t = \sigma dW_t$ was used in the third step of (29). In the last step we applied Theorem 4.4.9 from Shreve [22], which states that the Itô integral $I(t) = \int_0^t b(u)dW_u$ is normally distributed with expectation zero and variance $\int_0^t b^2(u)du$ for deterministic functions b. □

Let us now define

$$J\left(X_t, D_t, S_t, t, X_{[t,T]}\right) := \mathbb{E}\left[\widetilde{C}_t^{\Theta}|\mathcal{F}_t\right] + \frac{1}{2}a\ \operatorname{Var}\left(\widetilde{C}_t^{\Theta}|\mathcal{F}_t\right).$$

Then we obtain the following form of the optimisation problem (28):

$$C\left(X_t, D_t, S_t, t\right) = \min_{\Theta} J\left(X_t, D_t, S_t, t, X_{[t,T]}\right). \qquad (30)$$

It has been highlighted here that the process C, which we want to determine, depends on (X_t, D_t, S_t, t).

For an arbitrary feasible strategy $X_{[0,T]}$ and an arbitrary point in time $\hat{t} \in [0,T]$ we can define the composite strategy $X_{[0,\hat{t})} + X^*_{[\hat{t},T]}$: Until time $\hat{t}$ we buy shares as given by the arbitrary strategy and it is acted according to the optimal strategy afterwards where $X^*_{\hat{t}} = X_{\hat{t}}$ due to the left-continuity of the X-processes.

We thus have to determine C such that the following two conditions hold:

$$J\left(X_0, D_0, S_0, 0, X_{[0,\hat{t})} + X^*_{[\hat{t},T]}\right) \geq C(X_0, D_0, S_0, 0) \quad \text{for all } \hat{t}, X_{[0,\hat{t})} \tag{31}$$

$$J\left(X_0, D_0, S_0, 0, X^*_{[0,\hat{t})} + X^*_{[\hat{t},T]}\right) = C\left(X_0, D_0, S_0, 0\right) \quad \text{for all } \hat{t}. \tag{32}$$

Let us therefore have a closer look at the connection between the terms J and C in the following lemma.

Lemma 17 (J and C)**.** *Let us assume that the function C is sufficiently regular or more precisely $C(x,d,s,t) \in C^{2,1}(\mathbb{R}^3 \times [0,T]; \mathbb{R})$. Then*

$$\begin{aligned}
& J\left(X_0, D_0, S_0, 0, X_{[0,\hat{t})} + X^*_{[\hat{t},T]}\right) = C\left(X_0, D_0, S_0, 0\right) \\
+ \quad & \mathbb{E}\left[\int_0^{\hat{t}} \left(S_u + \frac{z}{2} + \lambda\left(X_0 - X_u\right) + D_u - C_X + \kappa C_D\right) \mu_u du\right] \\
+ \quad & \mathbb{E}\left[\int_0^{\hat{t}} \left(C_t - \rho D_u C_D + \frac{1}{2}\sigma^2 C_{SS} + \frac{1}{2} a \sigma^2 X_u^2\right) du\right] \\
+ \quad & \mathbb{E}\left[\sum_{u \in \mathcal{T}, u < \hat{t}} \left(\Delta C + \left(S_u + \frac{z}{2} + \lambda\left(X_0 - X_u\right) + D_u + \frac{1}{2q} x_u\right) x_u\right)\right] \\
=: \quad & C\left(X_0, D_0, S_0, 0\right) + I_1 + I_2 + I_3,
\end{aligned}$$

where $\Delta C(X_u, D_u, S_u, u) := C(X_u - x_u, D_u + \kappa x_u, S_u, u) - C(X_u, D_u, S_u, u)$ and C_X, C_D and C_t denote the first derivatives of the function C with respect to the processes X, D and the time. Analogously C_{SS} is the second derivative of C with respect to S.

Proof: We want to compute the function J of the composite strategy. To do this, we add the cost arising from the arbitrary strategy until $\hat{t}$ and the cost resulting from the optimal strategy from $\hat{t}$ onwards:

$$\begin{aligned}
& J\left(X_0, D_0, S_0, 0, X_{[0,\hat{t})} + X^*_{[\hat{t},T]}\right) = \\
& \mathbb{E}\left[\int_0^{\hat{t}} \left[\left(S_u + \frac{z}{2}\right) + \lambda\left(X_0 - X_u\right) + D_u\right] \mu_u du\right] + \\
& \mathbb{E}\left[\sum_{u \in \mathcal{T}, u < \hat{t}} \left[\left(S_u + \frac{z}{2}\right) + \lambda\left(X_0 - X_u\right) + D_u + \frac{1}{2q} x_u\right] x_u\right] + \\
& \frac{1}{2} a \int_0^{\hat{t}} \sigma^2 X_u^2 du + \mathbb{E}\left[C(X_{\hat{t}}, D_{\hat{t}}, S_{\hat{t}}, \hat{t})\right].
\end{aligned} \tag{33}$$

We used Lemma 16 for the computation of the variance.
We now want to apply the Itô formula to the function $C(X_t, D_t, S_t, t)$. In doing so, we have to respect that the process X has jumps. On account of this we apply the

Itô formula in the form of [21], Theorem 1.16 and therefore, need the regularity of C postulated in Lemma 17.

As it emanates from Definition 12 and (25), we have the following dynamic for X, D and S respectively:

$$\begin{aligned} X_t - X_0 &= -\int_0^t \mu_u du - \sum_{u \in \mathcal{T}, u<t} x_u \\ D_t - D_0 &= \int_0^t \left(-\rho D_u + \kappa \mu_u\right) du + \kappa \sum_{u \in \mathcal{T}, u<t} x_u \\ S_t - S_0 &= \sigma W_t. \end{aligned}$$

Hence the Itô formula for jump processes tells us

$$\begin{aligned} &C\left(X_t, D_t, S_t, t\right) - C\left(X_0, D_0, S_0, 0\right) = \\ &\int_0^t C_t du - \int_0^t \mu_u C_X du + \int_0^t C_D \left(-\rho D_u + \kappa \mu_u\right) du + \\ &\int_0^t C_S \sigma dW_u + \frac{1}{2}\sigma^2 \int_0^t C_{SS} du + \sum_{u \in \mathcal{T}, u<t} \Delta C. \end{aligned} \tag{34}$$

There are no C_{XX} or C_{DD} terms appearing in (34), since X and therefore D as well are deterministic.
The term (34) can now be plugged into (33) to finally get

$$\begin{aligned} & J\left(X_0, D_0, S_0, 0, X_{[0,\hat{t})} + X^*_{[\hat{t},T]}\right) = C\left(X_0, D_0, S_0, 0\right) \\ + \;& \mathbb{E}\left[\int_0^{\hat{t}} \left(S_u + \frac{z}{2} + \lambda\left(X_0 - X_u\right) + D_u - C_X + \kappa C_D\right) \mu_u du\right] \\ + \;& \mathbb{E}\left[\int_0^{\hat{t}} \left(C_t - \rho D_u C_D + \frac{1}{2}\sigma^2 C_{SS} + \frac{1}{2} a \sigma^2 X_u^2\right) du\right] \\ + \;& \mathbb{E}\left[\sum_{u \in \mathcal{T}, u<\hat{t}} \left(\Delta C + \left(S_u + \frac{z}{2} + \lambda\left(X_0 - X_u\right) + D_u + \frac{1}{2q} x_u\right) x_u\right)\right]. \end{aligned}$$

□

The remaining proof is organised as follows:
In Step I we construct the optimal strategy $X^*_{[0,T]}$ heuristically by using Lemma 17 and the conditions (31) and (32). In Step II we verify the optimal strategy we obtained in Step I and finally analyse it in detail in Step III.

Step I: Heuristic derivation of the optimal strategy
We make the following approach for the optimal cost, which suggests itself from the discrete time case.

$$C\left(X_t, D_t, S_t, t\right) = \left(S_t + \frac{z}{2}\right) X_t + \lambda X_0 X_t + \alpha_t X_t^2 + \beta_t X_t D_t + \gamma_t D_t^2$$

We want to specify α, β and γ so that the conditions (31) and (32) are satisfied. Thus we have, according to Lemma 17, to make sure that $I_1 + I_2 + I_3 \geq 0$ for every arbitrary strategy $X_{[0,\hat{t})}$ and $I_1 + I_2 + I_3 = 0$ if the arbitrary strategy is the optimal one. Fortunately the term I_3 does not make any difficulties, since it is always positive as one understands as follows:

$$\begin{aligned} -\Delta C &= C\left(X_u, D_u, S_u, u\right) - C\left(X_u - x_u, D_u + \kappa x_u, S_u, u\right) \\ &\leq \left(S_u + \frac{z}{2} + \lambda\left(X_0 - X_u\right) + D_u + \frac{1}{2q} x_u\right) x_u, \end{aligned} \tag{35}$$

where the last inequality is an equality in the case that $X^*_{[u,T]}$ has a jump in u. Therefore, it is sufficient to consider the terms I_1 and I_2 in the following. Defining

$$M_1 := S_t + \frac{z}{2} + \lambda\left(X_0 - X_t\right) + D_t - C_X + \kappa C_D \quad \text{and}$$

$$M_2 := C_t - \rho D_t C_D + \frac{1}{2}\sigma^2 C_{SS} + \frac{1}{2} a\sigma^2 X_t^2,$$

we have

$$\begin{aligned} M_1 &= S_t + \frac{z}{2} + \lambda\left(X_0 - X_t\right) + D_t \\ &- \left(S_t + \frac{z}{2} + \lambda X_0 + 2\alpha_t X_t + \beta_t D_t\right) + \kappa\left(\beta_t X_t + 2\gamma_t D_t\right) \\ &= \left(-2\alpha_t - \lambda + \kappa\beta_t\right) X_t + \left(1 - \beta_t + 2\kappa\gamma_t\right) D_t. \end{aligned}$$

That is M_1 is equal to zero when we set $\beta_t = f(t)$, $\alpha_t = \frac{\kappa f(t) - \lambda}{2}$ and $\gamma_t = \frac{f(t)-1}{2\kappa}$ for a function f that we still have to choose. Using this result to determine M_2 we get

$$M_2 = \frac{1}{2}\left[\kappa f'(t) + a\sigma^2\right] X_t^2 + \left[f'(t) - \rho f(t)\right] X_t D_t + \frac{1}{2\kappa}\left[f'(t) + 2\rho - 2\rho f(t)\right] D_t^2.$$

It is now our intention to specify f such that M_2 is always non-negative and equals zero for the optimal strategy. Hence we minimise M_2 in terms of X_t and we get the following candidate for the optimal strategy $X^*_{[0,T]}$

$$\begin{aligned} \frac{dM_2}{dX_t} &= \left[\kappa f'(t) + a\sigma^2\right] X_t + \left[f'(t) - \rho f(t)\right] D_t \overset{!}{=} 0 \Leftrightarrow \\ X_t^* &= \frac{-\left(f'(t) - \rho f(t)\right)}{\kappa f'(t) + a\sigma^2} D_t. \end{aligned} \tag{36}$$

We want that X^* plugged into M_2 gives zero. As an easy calculation shows, this will be the case if f satisfies the following Riccati differential equation

$$f'(t)\left(2\kappa\rho + a\sigma^2\right) - \kappa\rho^2 f^2(t) - 2a\sigma^2\rho f(t) + 2a\sigma^2\rho = 0.$$

Since $C(X_T, D_T, S_T, T) = [\left(S_T + \frac{z}{2}\right) + \lambda(X_0 - X_T) + D_T + \frac{1}{2q} X_T] X_T$ we have to respect the terminal condition $f(T) = 1$. The function

$$f(t) = \frac{v - a\sigma^2}{\kappa\rho} + \left[\left(\frac{\kappa\rho}{2v} - \frac{\kappa\rho}{v - a\sigma^2 - \kappa\rho}\right) e^{\frac{2\rho v(T-t)}{2\kappa\rho + a\sigma^2}} - \frac{\kappa\rho}{2v}\right]^{-1}$$

with $v := \sqrt{a^2\sigma^4 + 2a\sigma^2\kappa\rho}$ solves the considered differential equation. As stated in Proposition 13, we have $\lim_{a\to 0} f(t) = \frac{2}{\rho(T-t)+2}$, since

$$\begin{aligned}
&\lim_{a\to 0}\left[(\frac{\kappa\rho}{2v} - \frac{\kappa\rho}{v - a\sigma^2 - \kappa\rho})e^{\frac{2\rho v(T-t)}{2\kappa\rho+a\sigma^2}} - \frac{\kappa\rho}{2v}\right] = \\
&\lim_{a\to 0}\left[\frac{1}{2}\rho\frac{\kappa}{v}(e^{\frac{2\rho v(T-t)}{2\kappa\rho+a\sigma^2}} - 1) + 1\right] = \frac{1}{2}\rho(T-t) + 1.
\end{aligned}$$

Step II: Verification of the heuristic derivation
Let us first check that the function $C(x,d,s,t) = (s+\frac{z}{2})x + \lambda X_0 x + [\alpha_t x^2 + \beta_t xd + \gamma_t d^2]$ with α, β and γ from Step I, as given in Proposition 13 and 15 respectively, is sufficiently regular in order to justify the use of the Itô formula in Lemma 17 ex post. In other words, C has to be continuous differentiable in t and in addition two times continuous differentiable in all other three components. Hence, we will have to check if α, β and γ are continuous differentiable.
In the situation of Proposition 13 we have

$$\alpha_t' = \frac{\rho\kappa}{[\rho(T-t)+2]^2}, \quad \beta_t' = \frac{2\rho}{[\rho(T-t)+2]^2} \quad \textit{and} \quad \gamma_t' = \frac{\rho}{\kappa\,[\rho(T-t)+2]^2}.$$

That is α, β and γ are continuous differentiable, since $\rho(T-t)+2 = 0$ if and only if $t = T + \frac{2}{\rho} > T$.

In the situation of Proposition 15 we obtain

$$\alpha_t' = \frac{1}{2}\kappa f'(t), \quad \beta_t' = f'(t) \quad \textit{and} \quad \gamma_t' = \frac{1}{2\kappa} f'(t).$$

The Riccati differential equation tells us

$$f'(t) = \frac{1}{2\kappa\rho + a\sigma^2}\left(\kappa\rho^2 f^2(t) + 2a\sigma^2\rho(f(t)-1)\right)$$

and consequently it is only left to consider the continuity of the function f in Proposition 15. We know that

$$\left(\frac{\kappa\rho}{2v} - \frac{\kappa\rho}{v - a\sigma^2 - \kappa\rho}\right)\exp\left(\frac{2\rho v(T-t^*)}{2\kappa\rho + a\sigma^2}\right) - \frac{\kappa\rho}{2v} =: c_1 \exp\left(\frac{2\rho v(T-t^*)}{2\kappa\rho + a\sigma^2}\right) - c_2 = 0$$

if and only if $t^* = T - \frac{ln(\frac{c_2}{c_1})(2\kappa\rho + a\sigma^2)}{2\rho v}$. That is, $t^* > T$ if $\frac{c_2}{c_1} < 1$. Due to

$$\frac{c_2}{c_1} = \frac{v - a\sigma^2 - \kappa\rho}{-v - a\sigma^2 - \kappa\rho} < \frac{\kappa\rho - \sqrt{\kappa\rho}}{\kappa\rho + \sqrt{\kappa\rho} + 2a\sigma^2},$$

f is continuous, as desired, if $\kappa\rho > a^2\sigma^4$. This inequality will hold if the parameters are reasonably chosen.

Let us now verify again that our C, in fact, satisfies the conditions (31) and (32). As just explained, C is sufficiently regular and Lemma 17 can be applied. Through our choice of α, β and γ, M_1 and therefore I_1 are equal to zero. Furthermore, it can be recalculated that

$$\frac{d^2M_2}{dX_t^2} = \kappa f'(t) + a\sigma^2 > 0.$$

Hence, M_2 really has a minimum in $X_t = g(t)D_t$ and f has been chosen, such that M_2 is zero for this X. Therefore, I_2 is zero for the optimal strategy and positive for all others. The same holds for I_3 as mentioned in Step I. Consequently C satisfies the optimality conditions (31) and (32).

Step III: Analysis of the optimal strategy
In the last step of this proof we want to analyse the optimal strategy (36) in more detail.

At first we consider the strategy for $a = 0$ where we can write

$$\begin{aligned} X_t &= \frac{\rho f(t) - f'(t)}{\kappa f'(t)} D_t = \frac{1}{\kappa} D_t \left(\frac{\rho f(t)}{f'(t)} - 1 \right) \\ &= \frac{1}{\kappa} D_t \left(\frac{2\rho}{\rho(T-t)+2} \frac{[\rho(T-t)+2]^2}{2\rho} - 1 \right) = \frac{\rho(T-t)+1}{\kappa} D_t. \end{aligned} \tag{37}$$

Therefore, we get for the shares to be bought at time $t = 0$ as desired

$$X_0 - x_0 = X_{0+} = \frac{\rho T + 1}{\kappa} D_{0+} = \frac{\rho T + 1}{\kappa} \kappa x_0 \Leftrightarrow x_0 = \frac{X_0}{\rho T + 2}.$$

Let us now determine the buying intensity. By using (37), the product rule and

$$dD_t = -\rho D_t dt + \kappa \mu_t dt, \tag{38}$$

we get

$$dX_t = -\frac{\rho}{\kappa} D_t dt + \frac{\rho(T-t)+1}{\kappa} (\kappa \mu_t - \rho D_t) dt.$$

Setting $dX_t = -\mu_t dt$, we can derive

$$\mu_t = \frac{\rho}{\kappa} D_t.$$

According to (38) this means

$$dD_t = 0.$$

That implies that, analogously to the discrete time case, the deviation between the intrinsic and the actual best ask price is constant and equal to $x_0\kappa = \frac{\kappa X_0}{\rho T+2}$ for all $t \in (0, T)$. This insight leads to

$$\mu_t = \frac{\rho}{\kappa} \frac{\kappa X_0}{\rho T + 2} = \frac{\rho X_0}{\rho T + 2}.$$

Because of D_t being constant, X_t is linear in t on $(0, T)$ according to (38). Except for $t = 0$ and $t = T$, there are no further discrete trades. For the shares to be bought in T we readily conclude

$$X_0 = \frac{X_0}{\rho T + 2} + \frac{\rho X_0}{\rho T + 2} T + x_T \Leftrightarrow x_T = \frac{X_0}{\rho T + 2}.$$

In the end, the optimal strategy for $a \neq 0$ remains to be considered. According to (36) we have

$$X_t = g(t)D_t \quad \text{with } g(t) := \frac{\rho f(t) - f'(t)}{\kappa f'(t) + a\sigma^2}.$$

One can check that g is continuous in case of $\kappa\rho > a^2\sigma^4$ (analogously to the continuity of f in Step II). For x_0 we have the following relation

$$X_0 - x_0 = g(0+)D_{0+} \Leftrightarrow x_0 = X_0 \frac{\kappa f'(0) + a\sigma^2}{\kappa\rho f(0) + a\sigma^2}.$$

Analogously to the case $a = 0$, one obtains by using the product rule

$$\mu_t = \frac{\rho g(t) - g'(t)}{1 + \kappa g(t)} D_t.$$

We have $D_t = \kappa x_0 \exp(-\int_0^t \frac{\kappa g'(u)+\rho}{1+\kappa g(u)} du)$, since D follows the dynamic

$$\begin{aligned} D_{0+} &= \kappa x_0 \text{ and} \\ dD_t &= \kappa\mu_t dt - \rho D_t dt = \left(\kappa \frac{\rho g(t) - g'(t)}{1 + \kappa g(t)} - \rho\right) D_t dt = -\frac{\kappa g'(t) + \rho}{1 + \kappa g(t)} D_t dt. \end{aligned}$$

This proves Proposition 13 and 15.

Remark 18. (Time-consistency of the optimal strategy for $a = 0$)
Here we want to examine if the optimal strategy obtained in Proposition 13 and 15 for risk aversion $a = 0$ is **time-consistent**. This will be relevant in Chapter 7 when we examine call auctions.
Time-consistency means that the optimisation in an arbitrary point in time $\hat{t} \in (0, T)$ will not change the optimal strategy calculated in $t = 0$ if all parameters stay the same. To do so, we assume that an arbitrary strategy is used until $\hat{t}$ and examine how the optimal strategy from $\hat{t}$ onward looks like. This means that we determine the size of the discrete trade $\tilde{x}_{\hat{t}}$ in $\hat{t}$. In Step III of the proof of Proposition 15 we have seen that the optimal strategy is given by $X_t = \frac{\rho(T-t)+1}{\kappa} D_t$. Therefore one obtains:

$$\begin{aligned} X_{\hat{t}} - \tilde{x}_{\hat{t}} &= X_{\hat{t}+} = \frac{\rho(T-\hat{t})+1}{\kappa} D_{\hat{t}+} = \frac{\rho(T-\hat{t})+1}{\kappa} (D_{\hat{t}} + \kappa \tilde{x}_{\hat{t}}) \\ \Leftrightarrow \tilde{x}_{\hat{t}} &= \frac{1}{\rho(T-\hat{t})+2} \left(X_{\hat{t}} - \frac{\rho(T-\hat{t})+1}{\kappa} D_{\hat{t}}\right). \end{aligned} \tag{39}$$

If one buys optimally until $\hat{t}$, then

$$X_{\hat{t}} = X_0 - x_0 - \int_0^{\hat{t}} \mu_t dt = X_0 \frac{\rho(T-\hat{t})+1}{\rho T + 2}$$

and because of $dD_t = 0$ we have $D_{\hat{t}} = \kappa x_0 = \kappa \frac{X_0}{\rho T+2}$. Consequently, we get $\tilde{x}_{\hat{t}} = 0$. Therefore, the optimal strategy is time-consistent, since the buying intensity stays constant from $\hat{t}$ onward, too:

$$\tilde{\mu}_t = \frac{\rho}{\kappa} D_t = \frac{\rho X_0}{\rho T + 2}.$$

It is interesting that we can even use (39) to compute $\tilde{x}_{\hat{t}}$, when arbitrary $X_{\hat{t}}$ and $D_{\hat{t}}$ are given. E.g. buying arbitrary $y \in \mathbb{N}$ shares at $t = 0$ and not doing any other trades until $\hat{t}$, we receive that it is optimal to purchase

$$\tilde{x} = \frac{1}{\rho(T-\hat{t})+2}(X_0 - y[1 + (\rho(T-\hat{t})+1)e^{-\rho\hat{t}}]).$$

shares at $\hat{t}$. Corresponding to intuition, this term will become negative if y is especially large and $\hat{t}$ quite small. It makes also sense in case of having $y = 0$. Besides it is remarkable that $\tilde{D}_t = \kappa x$, $\tilde{\mu}_t = \rho x$ for $t \in (\hat{t}, T]$ and $\tilde{x}_T = x$ where

$$x := ye^{-\rho\hat{t}} + \tilde{x}_{\hat{t}}.$$

Incidentally, an analogous result will hold if there are only discrete trades x_i in $t_i < \hat{t}$ until $\hat{t}$.

4.3 Comparison with the simplest trading strategy

At the end of this chapter we compare the cost of the trading strategy obtained in Proposition 13 with the strategy which we call the simple Almgren and Chriss strategy with $x_0 = x_T = 0$ and buying intensity $\mu^* = \frac{X_0}{T}$. In case of this strategy, we have the following expected cost when we assume a best ask price as explained in (24):

$$C_0^* = \mathbb{E}\left[\int_0^T A_t^* \mu^* dt\right] = \mathbb{E}\left[\int_0^T \left[\left(S_t + \frac{z}{2}\right) + \lambda X_0 \frac{t}{T} + D_t^*\right] \frac{X_0}{T} dt\right].$$

Thereby D^* follows the dynamic $dD_t^* = (\kappa\mu^* - \rho D_t^*)dt$ and consequently we have

$$D_t^* = \frac{\kappa X_0}{\rho T}\left(1 - e^{-\rho t}\right).$$

The above formula simplifies to

$$C_0^* = \left(S_0 + \frac{z}{2}\right) X_0 + \left(\frac{\lambda}{2} + \frac{\rho T - \left(1 - e^{-\rho T}\right)}{(\rho T)^2}\kappa\right) X_0^2.$$

In comparison to

$$C_0 = \left(S_0 + \frac{z}{2}\right) X_0 + (\lambda + \alpha_0) X_0^2 = \left(S_0 + \frac{z}{2}\right) X_0 + \left(\frac{\lambda}{2} + \frac{\kappa}{\rho T + 2}\right) X_0^2$$

from Proposition 13, this gives an always positive deviation between C_0^* and C_0 of

$$C_0^* - C_0 = \frac{2\rho T - \left(1 - e^{-\rho T}\right)(\rho T + 2)}{(\rho T)^2(\rho T + 2)}\kappa X_0^2.$$

Comparing this cost difference $C_0^* - C_0$ with the total liquidity cost of the simple Almgren and Chriss strategy amounting to $C_0^* - \left(S_0 + \frac{z}{2}\right) X_0$, yields the following net deviation:

$$\Delta := \frac{C_0^* - C_0}{C_0^* - \left(S_0 + \frac{z}{2}\right) X_0}.$$

ρ	Half-life	$\lambda = \frac{1}{2q}$	$\lambda = \frac{1}{10q}$	$\lambda = \frac{1}{50q}$	$\lambda = 0$
0.001	693 days	0.01%	0.01%	0.02%	0.02%
0.5	1.39 days	2.82%	4.97%	5.86%	6.13%
1	270 min	3.98%	7.38%	8.91%	9.39%
2	135 min	4.32%	8.81%	11.14%	11.92%
5	54 min	2.64%	6.69%	9.66%	10.86%
20	13.5 min	0.37%	1.39%	3.03%	4.31%
50	5.4 min	0.07%	0.31%	0.93%	1.88%
1000	0.3 min	0.00%	0.00%	0.00%	0.10%

Table 2: Comparison of the optimal and the simple Almgren and Chriss strategy. Listed is Δ in percent for the different values of ρ and λ. The other parameters are chosen as usual.

Table 2 contains some sampled data for Δ. Setting e.g. $X_0 = 100{,}000$ shares, $\lambda = \kappa = \frac{1}{2q}$ with $q = 5{,}000$ shares and $T = 1$ day, then $C_0 - \left(S_0 + \frac{z}{2}\right) X_0$ lies in the dimension of 500,000 to 1 million Euro depending on the choice of ρ. Hence the difference to the simple Almgren and Chriss strategy, depending on the parameter choice, is not irrelevant but also not ground-breaking. A more detailed analysis of this issue does not seem too important, considering in practice not only the minimisation of the expectation of the cost but also risk has to be borne in mind.

We have seen in this chapter that the results achieved in Section 3 can be perfectly generalised to continuous trading time. Moreover, the optimal strategy for the risk-neutral case is time-consistent and its expected cost has been compared to the simple Almgren and Chriss strategy.
So far this diploma thesis was primarily a repetition of the work of Obizhaeva and Wang [20], but we improved and extended the representation as well as the results. In contrast, the material provided below represents new results that can not be found in the literature as of yet.

5 Time dependent parameters

In this chapter we start to discuss possible extensions of the LOB model. We will generalise it by replacing constant parameters by deterministic functions. As stated in the Obizhaeva and Wang paper [20], it is for instance possible to model the resiliency ρ as being time dependent, instead of a constant. This seems reasonable facing the U-shaped pattern during the trading day of the spread, the price volatility and the trading volume observed in a lot of market places (see e.g. the paper of Admati and Pfleiderer [1] for details). We will shortly examine this aspect in Subsection 5.2.
In the next subsection, we similarly want to take the market depth q as being time dependent. This might for example be useful when modelling the closing of the New York Stock Exchange by a peak of the market depth, since the traded volume is higher at this time.

5.1 Intraday curves of the market depth

We consider one trading day with discrete trading times $(t_n)_{n=0,...,N}$ and distance

$$\tau = t_{n+1} - t_n = \frac{1}{N}.$$

Instead of assuming a constant market depth q during the whole day, we allow q to be time dependent. This results in a given positive, deterministic sequence $(q_n)_{n=0,...,N}$. For example, the market depth might follow a U-shaped pattern during the trading day just as the traded volume. We assume that the ratio of permanent to total impact is a constant $\widehat{\lambda} \in [0,1]$ and hence the ratio of temporary to total impact is $\widehat{\kappa} := 1 - \widehat{\lambda}$. This indicates that the trade x_0 causes a permanent price impact of $\frac{\widehat{\lambda}}{q_0} x_0$ and a temporary price impact of $\frac{\widehat{\kappa}}{q_0} x_0$. Therefore, our permanent price impact at time t can not be determined by $\lambda(X_0 - X_t)$ for a constant λ anymore. We need to invent a new process D^p which represents the permanent impact. Its dynamic is similar to the temporary impact, which we will call D^t in the following.

$$\begin{aligned} D_0^p &= 0 \;\text{ and }\; D_{t_n}^p = D_{t_{n-1}}^p + \frac{\widehat{\lambda}}{q_{n-1}} x_{n-1} \quad \text{for } n = 1, ..., N \\ D_0^t &= 0 \;\text{ and }\; D_{t_n}^t = \left(D_{t_{n-1}}^t + \frac{\widehat{\kappa}}{q_{n-1}} x_{n-1} \right) e^{-\rho\tau} \quad \text{for } n = 1, ..., N \end{aligned}$$

Accordingly the average price per share at time t_n is

$$\overline{P}_{t_n} = \left(S_{t_n} + \frac{z}{2} \right) + D_{t_n}^p + D_{t_n}^t + \frac{x_n}{2q_n}.$$

With these assumptions we acquire the following result for the optimal strategy whose proof is again a straight forward backward induction as in Lemma 4.

Corollary 19. *(Optimal trading strategy for time dependent q)*
The expected cost under the optimal strategy is

$$C_{t_n} = \left(S_{t_n} + \frac{z}{2} \right) X_{t_n} + X_{t_n} D_{t_n}^p + \left[\alpha_n X_{t_n}^2 + \beta_n X_{t_n} D_{t_n}^t + \gamma_n (D_{t_n}^t)^2 \right].$$

The associated optimal strategy is given by $x_N = X_{t_N}$ *and*

$$x_n = \frac{1}{2}\delta_{n+1}\left[\epsilon_{n+1}X_{t_n} - \phi_{n+1}D^t_{t_n}\right] \quad \textit{for } n = 0, ..., N-1,$$

where we used the following recursive sequences

$$\begin{aligned}
\alpha_N &= \frac{1}{2q_N} \quad \textit{and} \quad \alpha_n = \alpha_{n+1} - \frac{1}{4}\delta_{n+1}\epsilon^2_{n+1} \\
\beta_N &= 1 \quad \textit{and} \quad \beta_n = \beta_{n+1}e^{-\rho\tau} + \frac{1}{2}\delta_{n+1}\epsilon_{n+1}\phi_{n+1} \\
\gamma_N &= 0 \quad \textit{and} \quad \gamma_n = \gamma_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}\phi^2_{n+1} \\
\delta_n &= \left[\frac{1}{2q_{n-1}} - \frac{\widehat{\lambda}}{q_{n-1}} + \alpha_n - \frac{\widehat{\kappa}}{q_{n-1}}e^{-\rho\tau}\beta_n + (\frac{\widehat{\kappa}}{q_{n-1}})^2 e^{-2\rho\tau}\gamma_n\right]^{-1} \\
\epsilon_n &= 2\alpha_n - \frac{\widehat{\lambda}}{q_{n-1}} - \frac{\widehat{\kappa}}{q_{n-1}}e^{-\rho\tau}\beta_n \\
\phi_n &= 1 - e^{-\rho\tau}\beta_n + 2\frac{\widehat{\kappa}}{q_{n-1}}e^{-2\rho\tau}\gamma_n.
\end{aligned}$$

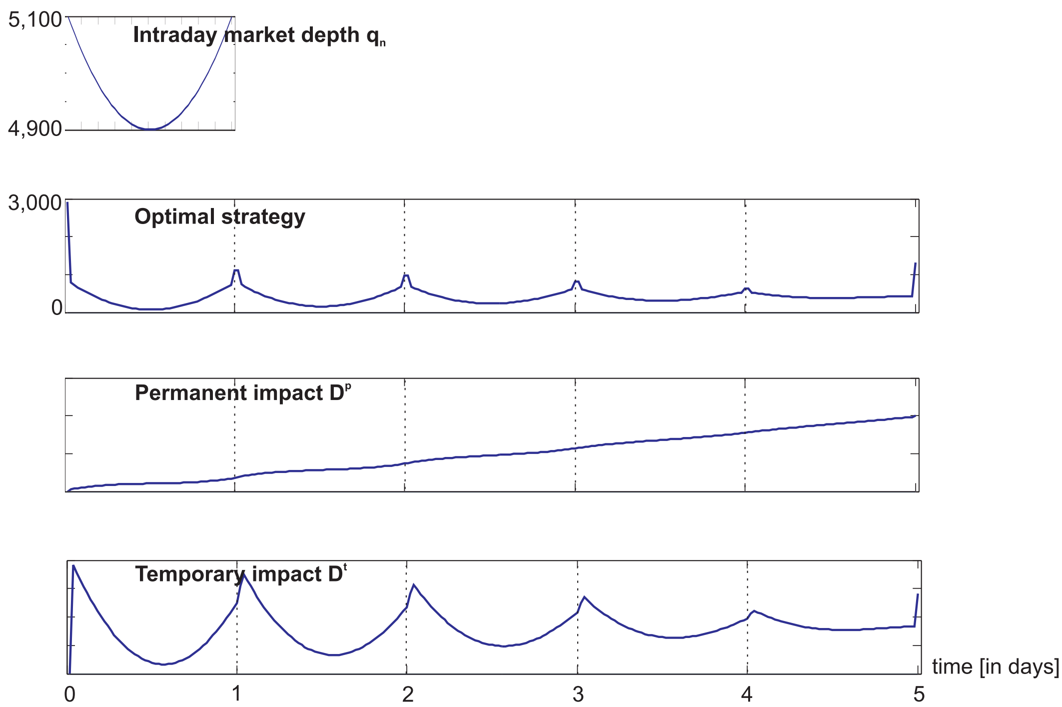


Figure 12: Optimal strategy with the associated permanent and temporary impact for q having the shape of a parabola with $\underline{q} = 4{,}900$ and $\overline{q} = 5{,}100$. We have chosen $X_0 = 100{,}000$ shares, $\rho = 20$, $\widehat{\lambda} = \widehat{\kappa} = \frac{1}{2}$ and considered a period of $d = 5$ days and $N = 49$ trading times per day.

Intuitively one might expect D^p to appear in the optimal x_n, but surprisingly the structure of this corollary is equivalent to that in Lemma 4. We only incorporate the fact that λ and κ are no constants anymore.

Calculating backwards the first entries of the sequences δ_n to ϕ_n, one recognizes that they depend non-trivially on $q_{n-1}, ..., q_N$. This means that every x_n of the optimal strategy depends on $q_0, ..., q_N$.
The corollary can easily be generalised to a trading horizon of several days $d \in \mathbb{N}$ with trading times $(t_{i,j})_{\substack{i=0,...,N \\ j=1,...,d}}$, each day having the same developing of the market depth $(q_n)_{n=0,...,N}$. Taking explicit values for q_n, we can calculate the optimal strategy.

Example 20. (Exemplary choices for the market depth) Take $\underline{q}, \overline{q} \in \mathbb{R}_{>0}$ with $\overline{q} > \underline{q}$.

1. **Straight line** $q_n = (\overline{q} - \underline{q})n\tau + \underline{q}$
2. **Parabola** $q_n = 4(\underline{q} - \overline{q})(n\tau - (n\tau)^2) + \overline{q}$
3. **Cosine** $q_n = \frac{1}{2}\left[(\underline{q} - \overline{q})\cos(2\pi n\tau) + \underline{q} + \overline{q}\right]$
4. **Randomly** $q_n = \underline{q} + (\overline{q} - \underline{q})r_n$ for independently and uniformly distributed random numbers $r_n \in [0, 1]$.

The correct model for the deterministic market depth (q_n) should be chosen according to the market place. If we usually experience a U-shaped pattern for the market depth, a parabola as in 2., with $\overline{q}$ shares of market depth at the boundaries of the day and $\underline{q}$ shares of market depth in the middle of the day will be reasonable. In a foreign exchange market where according to [12] trading occurs 24 hours a day and market depth peaks in the middle of the day, a cosine as in 3. might be befitting.

Calculating the optimal strategy for different choices of q, one observes x_n moving in the same direction as q_n. This tendency is displayed in Figure 12. Moreover, it is remarkable that the temporary impact D_n^t under the optimal strategy accompanies q_n as well. Note that it is not constant anymore! The permanent impact grows nearly linearly. Unfortunately, the parameters ρ etc. have to be chosen carefully. Otherwise negative values for some x_n can occur.

If one assumes the market depth q to be stochastic, e.g. a mean reverting stochastic process, instead of only deterministic, the dynamic programming method cannot be used to find the optimal strategy.

5.2 Intraday curves of the resiliency

In the following, we will use a constant q if not stated otherwise. Again, we consider one trading day with discrete trading times $(t_n)_{n=0,...,N}$ and distance

$$\tau = t_{n+1} - t_n = \frac{1}{N+1}.$$

Instead of taking a constant resiliency ρ of the LOB during the whole day, we allow ρ being time dependent, i.e. we have a given deterministic, integrable function $(\rho_t)_{t\in[0,T]}$. We can take similar functions as in Example 20.

Example 21. (Exemplary choices for the resiliency)
Take $\underline{\rho}, \overline{\rho} \in \mathbb{R}_{>0}$ with $\overline{\rho} > \underline{\rho}$.

1. **Straight line** $\rho_t = \left(\overline{\rho} - \underline{\rho}\right) \frac{t}{T} + \underline{\rho}$ $\qquad \int_{t_{n-1}}^{t_n} \rho_t dt = \underline{\rho}\tau + \frac{\overline{\rho}-\underline{\rho}}{2T}\tau^2(2n-1)$

2. **Parabola** $\rho_t = 4\left(\underline{\rho} - \overline{\rho}\right)\left(\frac{t}{T} - (\frac{t}{T})^2\right) + \overline{\rho}$
$$\int_{t_{n-1}}^{t_n} \rho_t dt = \overline{\rho}\tau + 4\left(\underline{\rho} - \overline{\rho}\right)\left[\frac{t_n^2 - t_{n-1}^2}{2T} - \frac{t_n^3 - t_{n-1}^3}{3T^2}\right]$$

3. **Cosine** $\rho_t = \frac{1}{2}\left[\left(\underline{\rho} - \overline{\rho}\right)\cos\left(2\pi\frac{t}{T}\right) + \underline{\rho} + \overline{\rho}\right]$
$$\int_{t_{n-1}}^{t_n} \rho_t dt = \frac{1}{2}\left[\left(\underline{\rho} + \overline{\rho}\right)\tau + \frac{T\left(\underline{\rho} - \overline{\rho}\right)}{2\pi}\left(\sin\left(2\pi\frac{t_n}{T}\right) - \sin\left(2\pi\frac{t_{n-1}}{T}\right)\right)\right]$$

Taking a time dependent resiliency does not change the structure of the backward induction as given in Lemma 4. We only have to replace $\rho\tau$ by $\int_{t_{n-1}}^{t_n} \rho_t dt$ in the induction step and the dynamic of D becomes

$$D_{t_n} = \left(D_{t_{n-1}} + \kappa x_{n-1}\right)\exp(-\int_{t_{n-1}}^{t_n} \rho_t dt).$$

We get similar results as in Subsection 5.1: Apart from the two block trades at 0 and T, the optimal strategy has the same developing as the chosen resiliency. This is illustrated in Figure 13. When we e.g. assume the resiliency to follow a cosine curve during each day, the optimal strategy is a cosine as well. Moreover, the trading profile of the optimal strategy does not change from day to day.

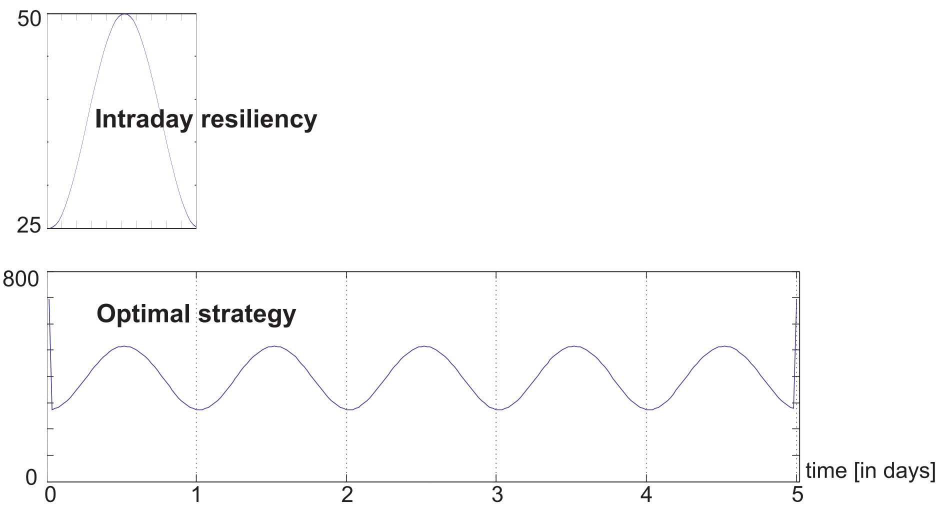


Figure 13: Optimal strategy for the resiliency ρ_t being a cosine with $\underline{\rho} = 25$ and $\overline{\rho} = 50$. We have chosen $X_0 = 100{,}000$ shares, $q = 5{,}000$, $\lambda = \frac{1}{2q}$ and considered a period of $d = 5$ days and $N = 49$ trading times per day.

5.3 Time-dependent market depth, resiliency, spread and volatility

To complete this section, we want to state that the optimal strategy cannot only be adjusted for time-dependent market depth and resiliency as presented in the Subsections 5.1 and 5.2, but the spread z and the volatility σ of the price process

$$S_t = S_0 + \int_0^t \sigma_s dW_s$$

can be assumed to follow a deterministic intraday curve as well. We suppose that we are given (q_n), (z_n) for $n = 0, ..., N$ and (ρ_t), (σ_t) for $t \in [0, T]$. Our average price at time t_n is

$$\overline{P}_{t_n} = \left(S_{t_n} + \frac{z_n}{2}\right) + D^p_{t_n} + D^t_{t_n} + \frac{x_n}{2q_n},$$

where the permanent impact process has the dynamic

$$D^p_0 = 0, \qquad D^p_{t_n} = D^p_{t_{n-1}} + \frac{\widehat{\lambda}}{q_{n-1}} x_{n-1}$$

and the temporary impact behaves according to

$$D^t_0 = 0, \qquad D^t_{t_n} = (D^t_{t_{n-1}} + \frac{\widehat{\kappa}}{q_{n-1}} x_{n-1}) \exp\left(-\int_{t_{n-1}}^{t_n} \rho_t dt\right).$$

Our optimisation problem is

$$C_{t_n} = \min_{\left\{x_n,...,x_N \in \mathbb{R} \mid \sum_{k=n}^N x_k = X_{t_n}\right\}} \left\{ \mathbb{E}\left[\sum_{k=n}^N \overline{P}_{t_k} x_k\right] + \frac{1}{2}\, a \operatorname{Var}\left(\sum_{k=n}^N \overline{P}_{t_k} x_k\right)\right\}.$$

Similar to Lemma 8 we can calculate the variance of our cost:

$$\operatorname{Var}\left(\sum_{k=n}^N \overline{P}_{t_k} x_k\right) = \sum_{k=n+1}^N \operatorname{Var}\left(S_{t_k} - S_{t_{k-1}}\right) X^2_{t_k} =$$
$$\sum_{k=n+1}^N \operatorname{Var}\left(\int_{t_{k-1}}^{t_k} \sigma_t dW_t\right) X^2_{t_k} = \sum_{k=n+1}^N \left(X^2_{t_k} \int_{t_{k-1}}^{t_k} \sigma^2_t dt\right).$$

Incorporating all this into our established backward induction gives the following result.

Corollary 22. *(Optimal trading strategy for time dependent q, ρ, z and σ)*
The expected cost under the optimal strategy is

$$C_{t_n} = S_{t_n} X_{t_n} + X_{t_n} D^p_{t_n} + \left[\alpha_n X^2_{t_n} + \beta_n X_{t_n} D^t_{t_n} + \gamma_n (D^t_{t_n})^2 + \eta_n X_{t_n} + \mu_n D^t_{t_n} + \omega_n\right].$$

The associated optimal strategy is given by $x_N = X_{t_N}$ and

$$x_n = \frac{1}{2}\delta_{n+1}\left[\epsilon_{n+1} X_{t_n} - \phi_{n+1} D^t_{t_n} - \varphi_{n+1}\right] \text{ for } n = 0, ..., N-1,$$

where the used sequences can be found in Appendix A.2.

The bottom line is that we can replace the constant parameters in our model by time dependent functions. But we have to be careful when we do so, because our optimal strategy might be partly negative.

6 Auctions–First approaches via volume time

So far we have only considered continuous trading and ignored the fact that periodic **call auctions** or sometimes referred to as clearing house auctions can be found in many market places. In this chapter, we therefore aim to present first ideas concerning these auctions.

There are different designs of call auctions. In most market places, a morning and evening call auction take place every trading day. Continuous trading is stopped for a few minutes while bidding continues. Limit and market orders flow into the order book without crossing of the matching orders. As a result, the sell and the buy side of the order book may start overlapping.
At the end of this order collection phase, one price P^* per share is determined by an algorithm such that the executable order volume is maximised. How this algorithm looks like, will be discussed later in Chapter 8.
Depending on its limit price p, a limit order in the auction is executed or not–in the case of a buy order it is executed at P^* if $p \geq P^*$ and vice versa for a sell order. Price-time priority rules are used as during continuous trading. Orders that cannot be conducted at the determined auction price P^* are transferred to the following continuous trading phase or call auction respectively.
This basic knowledge about call auctions is sufficient to go through the next two chapters.

Remark 23. Since exactly one price P^* is fixed, call auctions are sometimes referred to as single-price auctions in comparison to the continuous trading which is sometimes called double-price auction because both the best ask and bid price are stated.

Although an auction only takes five minutes physical time, it often constitutes more than 5% of the daily volume in a stock. This fact already emphasises the importance of call auctions. According to Kehr, Krahnen and Theissen, who investigate in [15] data of stocks in the DAX index from 1996, more than 20% of daily trading of the Siemens stock is carried out through auctions. This figure can even rise up to 50% in case of less liquid stocks and is on average roughly 10% of daily volume for each auction.
Our time horizon T often spans more than one day. That is why they have to be taken into account and cannot be disregarded as in most literature. To get a better insight into the subject of auctions, we will shortly mention how they blend in the trading of Frankfurt and London Stock Exchanges (FSE and LSE respectively), which are the leading markets in Europe.

Example 24. The order driven trading systems at the LSE and FSE are called SETS and Xetra respectively. Both have been operating since 1997. In case of the LSE, dealer quotes are still used in parallel. In Table 3 an overview of the occurring auctions is given. At the end of an auction, market order or price monitoring extensions may take place. This is the case if market orders are left unexecuted or the potential auction price is outside the price monitoring tolerance. On top of this, there is a 0 to 30 second random end. The idea of this random end is to avoid last-minute manipulation of the auction price.

Auctions	Morning[2]	Intraday	Evening
Duration	10 min	2 min	5 min
SETS	7.50 a.m.	none	4.30 p.m.
Xetra	8.50 a.m.	1 p.m.	5.30 p.m.

Table 3: Auctions in SETS and Xetra.

The algorithms that calculate the clearing price need less than 30 seconds. Execution acknowledgments are sent to the appropriate market participants after the auction.
In addition to the morning, intraday and evening auctions, one can find Automatic Execution Suspension periods of roughly five minutes duration. They occur during continuous trading if the potential execution price breaches the price tolerance levels. These breaks are called volatility breaks in the Xetra system and are handled similarly to auctions.
The question arises which information of the auction the traders can see during the calling phase. One extreme, which we will call **blind auctions**, are e.g. Chinese call auctions where no information is disseminated at all except of the final clearing price. The other extreme, so-called **visible auctions**, are e.g. auctions at the LSE where the whole LOB, or more exactly the ten ticks smaller than the best bid and larger than the best ask price, are publicly visible. In this sense the FSE is a hybrid: The indicative price with its associated trading volume and imbalance is observable to the market participants during the call phase, but not the individual orders. As the name suggests, the indicative price is the price per share which would result in a maximum tradeable volume if only the orders accumulated so far were incorporated.
According to Beltran-Lopez and Frey [4], the rational of closing the book during auctions is not clear, since the lack of transparency may harm the dissemination of information. The main argument is that by hiding the book, the exchange protects large orders.

6.1 Blind auctions with Almgren and Chriss model

In the following subsection, we aim to introduce a very basic approach to auctions in the simple Almgren and Chriss model from [3]. As a reminder, the model consists of a linear permanent price impact and an only instantaneously existing linear temporary impact. No delay effects are involved. Consequently the average price per share at time t_n is

$$\overline{P}_{t_n} = \left(S_{t_n} + \frac{z}{2}\right) + \lambda \sum_{i=0}^{n} x_i + \frac{\eta}{t_n - t_{n-1}} x_n \tag{40}$$

for constants $\lambda, \eta \in \mathbb{R}_{>0}$.
We will now model one trading day with a morning and evening auction of the same length. We do that by taking the **volume time**, instead of the physical time of the auctions. This means that the auction length v is chosen corresponding to the portion of daily trading volume. For simplicity, we norm the time axis by defining one time

[2] The morning auction includes orders from the post and pre-trading phase.

unit as the time of continuous trading between the morning and evening auction of one day. E.g. let us assume that one time unit is 6.5 hours and $y > 0$ is the average trading volume measured in shares during this continuous trading phase of one day. We refer to y_a as the average number of shares traded on one auction. The idea is to include the auctions in the setting of Almgren and Chriss by choosing their length as $v = \frac{y_a}{y}$. In the extreme case, for instance, where on average the same number of shares is traded during an auction and the continuous trading of one day, we would model v as 6.5 hours of normal trading. The real time length of five minutes is not decisive.

For the time being, we assume that the considered auctions are totally blind. Therefore, it does not matter how we phase our orders during the auction. Therefore, we only optimise over $x_0, ..., x_N \in \mathbb{R}$, where x_0 and x_N are the number of shares to be put on the morning and evening auction respectively and x_1 to x_{N-1} are the trades to be executed during continuous trading. The corresponding points in time are

$$t_0 = v,\ t_n = v + n\tau \text{ for } n = 1, ..., N-1 \text{ and } t_N = 2v + 1,$$

where $\tau := \frac{1}{N-1}$, $v \in \mathbb{R}_{>0}$ is fixed and t_{-1} which appears in (41) is set to zero. The same price behaviour as in (40) is presumed for the auction prices.

The following optimisation problem, which is matchable to (19) and (20), emerges when the expectation of our cost are meant to be minimised

$$\begin{aligned} C_0 &= \left(S_0 + \frac{z}{2}\right) X_0 + \frac{\lambda}{2} X_0^2 \\ &+ \min_{\left\{x_0,...,x_N \in \mathbb{R} \mid \sum_{n=0}^N x_n = X_0\right\}} \left\{ \frac{\lambda}{2} \sum_{n=0}^N x_n^2 + \eta \sum_{n=0}^N \frac{x_n^2}{t_n - t_{n-1}} \right\}. \end{aligned} \tag{41}$$

By defining $c_1 := \frac{\lambda}{2} + \frac{\eta}{\tau}$ and $c_2 := \frac{\lambda}{2} + \frac{\eta}{v}$ we have to consider

$$\begin{aligned} \min_{\left\{x_0,...,x_N \in \mathbb{R} \mid \sum_{n=0}^N x_n = X_0\right\}} &\left\{ c_1 \sum_{n=1}^{N-1} x_n^2 + c_2 \left(x_0^2 + x_N^2\right) \right\} = \\ \min_{\left\{x_0, x_1 \in \mathbb{R} \mid x_1 = \frac{X_0 - 2x_0}{N-1}\right\}} &\left\{ c_1 (N-1) x_1^2 + 2 c_2 x_0^2 \right\} = \min_{x_0 \in \mathbb{R}} f(x_0) \end{aligned}$$

with $f(x) := c_1 \frac{(X_0 - 2x)^2}{N-1} + 2c_2 x^2$. We know that the optimal x_n are equal for $n = 1, .., N-1$ because they occur quadratic in the sum to be minimised. Thus it is sufficient to optimise with respect to one parameter[3] which gives a minimum in

$$x_0 = X_0 \frac{c_1}{(N-1)c_2 + 2c_1} \quad \text{and} \quad x_1 = X_0 \frac{c_2}{(N-1)c_2 + 2c_1}.$$

In particular, the obtained optimal strategy is positive as desired and linear in X_0. Let us now turn to its interpretation in more detail: As shown in Figure 14, we are

[3] Incidentally, we will have to optimise over $i \in \mathbb{N}$ in lieu of one parameter if we have i different auction lengths $v_1, ..., v_i \in \mathbb{R}_{>0}$.

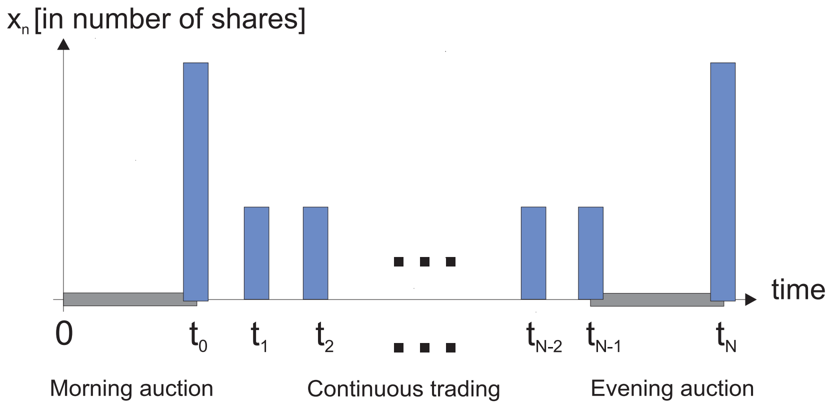


Figure 14: Optimal trading strategy when blind auctions of the volume time length v are included in the Almgren and Chriss model.

meant to put x_0 shares on each of the morning and evening auction and to allocate the remaining shares equally over the normal trading day, i.e. $x_1 = ... = x_{N-1}$. In the following, we assume that the number of trading times N is large in order to have a small τ. This provokes that we have a comparatively lower temporary impact on the auctions, since $v > \tau$. Consequently $x_0 > x_1$, but choosing e.g. $v = 2\tau$ gives $x_0 < 2x_1$. For a better understanding of the dependencies of x_0 and x_1 on the parameters λ, η, τ and v, we define the ratio of the number of stocks traded during the auctions to the total number of stocks as

$$R(\lambda, \eta, \tau, v) := \frac{2x_0}{X_0} = \frac{\lambda + 2\frac{\eta}{\tau}}{\frac{1}{\tau}\left(\frac{\lambda}{2} + \frac{\eta}{v}\right) + \lambda + 2\frac{\eta}{\tau}} \in (0, 1).$$

Calculating the derivatives of R gives

$$\begin{aligned}
\frac{\partial}{\partial \lambda} R(\lambda, \eta, \tau, v) &= -\frac{\eta(v - \tau)}{v\tau^2 d(\lambda, \eta, \tau, v)^2} < 0 \\
\frac{\partial}{\partial \eta} R(\lambda, \eta, \tau, v) &= \frac{\lambda(v - \tau)}{v\tau^2 d(\lambda, \eta, \tau, v)^2} > 0 \\
\frac{\partial}{\partial \tau} R(\lambda, \eta, \tau, v) &= \frac{\lambda(\lambda v + 2\eta)}{2v\tau^2 d(\lambda, \eta, \tau, v)^2} > 0 \\
\frac{\partial}{\partial v} R(\lambda, \eta, \tau, v) &= \frac{\eta(\lambda\tau + 2\eta)}{v^2\tau^2 d(\lambda, \eta, \tau, v)^2} > 0
\end{aligned}$$

Thereby $d(\lambda, \eta, \tau, v) := \frac{1}{\tau}\left(\frac{\lambda}{2} + \frac{\eta}{v}\right) + \lambda + 2\frac{\eta}{\tau}$ denotes the denominator of the ratio R. The signs of the above derivatives tell us that we are trading less on the auctions when the permanent impact constant λ is increasing. We are trading more when the temporary impact η, the length of the auctions v or the distance τ between two trading times during the continuous trading increases.

We now want to see what happens when we choose the trading times during the continuous trading $[v, v+1]$ being arbitrarily near to each other. Using $\tau = \frac{1}{N-1}$ the limit $N \to \infty$ gives

$$\lim_{N\to\infty} x_0 = X_0 \frac{1}{2 + \frac{\lambda}{2\eta} + \frac{1}{v}} < \frac{1}{2} X_0 \quad \text{and} \quad \lim_{N\to\infty} x_1 = 0.$$

This limit of x_0 is intuitive, since the number of traded shares on the auctions increases as the auction length v increases or the ratio of the permanent to the temporary impact $\frac{\lambda}{\eta}$ decreases.

The results can be generalised easily to $K \in \mathbb{N}$ auctions of the same length instead of two. They can be placed arbitrarily during the considered continuous trading time normed as one time unit. In particular, several trading days can be modelled. Analogously to the case with two auctions, one can compute that it is optimal to trade x_0 shares on each of the K auctions and x_1 shares at each trading time during continuous trading with

$$\begin{aligned} x_0 &= X_0 \frac{c_1}{(N+1-K)c_2 + Kc_1} \overset{N\to\infty}{\longrightarrow} X_0 \frac{1}{K + \frac{\lambda}{2\eta} + \frac{1}{v}} < \frac{1}{K} X_0 \quad \text{and} \\ x_1 &= X_0 \frac{c_2}{(N+1-K)c_2 + Kc_1} \overset{N\to\infty}{\longrightarrow} 0. \end{aligned}$$

6.2 Blind auctions with Obizhaeva and Wang model

Now a similar approach as in Subsection 6.1 is presented to include blind auctions in the model of Obizhaeva and Wang [20]. Again, we adjust our time discretisation by taking into account the volume time of the considered auctions. The same price behaviour as in [20] given explicitly in (43) is assumed. But in contrary to Subsection 6.1, we want to allow the morning and evening auctions to have different lengths v_1 and v_2, respectively. Actually, auctions often have different lengths. We did not allow this in Subsection 6.1 because this would have complicated the optimisation.
Suppose we are considering an execution that occupies $d \in \mathbb{N}$ complete trading days and we have $N+1$ trading times per day. That is we look at the matrix of trading times $(t_{i,j})_{\substack{i=0,\dots,N \\ j=1,\dots,d}}$ with the associated matrix of discrete trades

$$(x_{i,j})_{\substack{i=0,\dots,N \\ j=1,\dots,d}}.$$

We want to interpret $x_{0,j}$ and $x_{N,j}$ as the number of shares to be bought on the opening and the closing auctions respectively on the j-th day. According to this, $x_{1,j}$ and $x_{N-1,j}$ are the first and the last trade during the continuous trading phase of the j-th day. Since we model a blind auction, other market participants do not react on our trades on the auctions. Therefore, we set

$$t_{0,j} - t_{N,j-1} = v_1,\ t_{N,j} - t_{N-1,j} = v_2 \text{ and } t_{i,j} - t_{i-1,j} = \tfrac{1}{N-1} \text{ for } i = 1, ..., N-1 \quad (42)$$

which corresponds to trading at the end of the auction. That is in comparison to the visible auctions in the subsequent Chapter 7, there is no resiliency effect for the auction trade itself during the auction.

Let $(a_{i,j})_{\substack{i=0,\dots,N \\ j=1,\dots,d}}$ be a matrix. Then we denote with $a'_{i,j}$ the successor of the entry $a_{i,j}$:

$$a'_{i,j} = \begin{cases} a_{0,j+1} & \text{if } i = N \\ a_{i+1,j} & \text{otherwise} \end{cases}.$$

Days:	1	2	3	4
x_0	10.06	1.70	1.70	1.70
x_1	2.00	2.00	2.00	2.00
x_2	2.00	2.00	2.00	2.00
x_3	2.00	2.00	2.00	2.00
x_4	2.00	2.00	2.00	2.00
x_5	2.00	2.00	2.00	2.00
x_6	2.00	2.00	2.00	2.00
x_7	2.00	2.00	2.00	2.00
x_8	2.00	2.00	2.00	1.45
x_9	1.70	1.70	1.70	2.20
x_{10}	1.40	1.40	1.40	9.80

Table 4: Optimal strategy in discrete time for blind morning and evening auctions. We want to buy $X_0 = 100{,}000$ shares, consider $d = 4$ days and the number of discrete trades per day is $N = 10$. The market depth is $q = 5{,}000$ shares, we set the permanent price impact parameter to $\lambda = \frac{1}{2q}$ and the resiliency coefficient to $\rho = 2$. The morning and evening auction have the same length with $v_1 = v_2 = \frac{0.5}{6.5}$ (meaning that an auction is modelled as 30 minutes in comparison to the 6.5 hours of continuous trading during one day). The $x_{i,j}$ in the table are given in thousand shares.

For convenience we write the processes D, X, C, $\overline{P}$ and S as $D_{i,j}$ instead of $D_{t_{i,j}}$. With these notations in mind we assume the following price per share behaviour:

$$\overline{P}_{i,j} = (S_{i,j} + \frac{z}{2}) + \lambda\,(X_0 - X_{i,j}) + D_{i,j} + \frac{x_{i,j}}{2q}, \tag{43}$$

where we have as in [20] the following dynamic of the process D

$$D'_{i,j} = (D_{i,j} + \kappa x_{i,j})\, e^{-\rho\left(t'_{i,j} - t_{i,j}\right)}.$$

Now we can do exactly the same dynamic programming procedure as in Lemma 4 and get identical results with the only difference being that the δ, ϵ and ϕ terms have a $e^{-\rho v_1}$ and $e^{-\rho v_2}$ instead of $e^{-\rho\tau}$ in the recursion $i = 0$ and $i = N$ respectively. This follows from

$$\begin{aligned} D_{0,j} &= (D_{N,j-1} + \kappa x_{N,j-1})\, e^{-\rho v_1} \quad \text{and} \\ D_{N,j} &= (D_{N-1,j} + \kappa x_{N-1,j})\, e^{-\rho v_2}. \end{aligned}$$

Taking explicit values for our model parameters, we can once again calculate our optimal strategy as done in Table 4. In addition to the first trade $x_{0,0}$ on the first trading day and the last three trades $x_{N-2,d}$ to $x_{N,d}$ on the last day, our optimal strategy consists of the following discrete trades:
We have equal $x_{N-1,j}$ for $j = 1, ..., d-1$, equal $x_{N,j}$ for $j = 1, ..., d-1$ and equal $x_{0,j}$ for $j = 2, ..., d$. All other shares are spread evenly over the continuous trading phase.
If the morning and evening auctions have the same length ($v_1 = v_2$), the trades before the evening auction and on the morning auction will be equal in their size, i.e.

$$x_{N-1,j} = x_{0,j+1} \quad \text{for} \quad j = 1, ..., d-1.$$

Increasing the length of the morning (evening) auction v_1 (v_2) leads to an increase of $x_{0,j}$ ($x_{N-1,j}$) and $x_{N,j}$. This can be explained with (16), since v_1 appears in $D_{0,j}$ and $\delta_{0,j}$ and v_2 appears in $D_{N,j}$ and $\delta_{N,j}$.
Similar to the case without auctions, the process D is constant during the continuous trading and for each auction phase $[t_{N-1,j}, t_{0,j+1}]$ we have the following behaviour: The discrete trade $x_{N-1,j}$ lifts the process D, which settles down during the evening auction of length v_2 to be lifted again by $\kappa x_{N,j}$ because of the evening auction bid. Settling down again during the morning auction with volume time v_1, D rises to its original level because of $x_{0,j+1}$.

In this section it has been clarified what we mean by an auction. First ideas how to incorporate blind auctions into the Almgren and Chriss as well as Obizhaeva and Wang model using volume time for the auctions have been presented.

7 Visible auctions with Obizhaeva and Wang model

In the following chapter we want to consider **visible auctions**. That means that other market participants are able to react on our auction bids. Therefore, we model the resiliency for these bids to start immediately during the auction.
The price impact is determined in exactly the same way as before. The auction price P^* per share is computed by adding the permanent and the temporary price impact to $S + \frac{z}{2}$. Suppose that t_{start} and $t_{end} \in [0, T]$ are the beginning and the end of the considered auction. That means $t_{end} - t_{start} = v_1$ or v_2 depending on which auction is looked at (morning or evening auction). As explained in Chapter 6, volume time is used to model the auction lengths in order to account for the higher liquidity during auctions. It should be emphasised here that bidding is allowed on $[t_{start}, t_{end}]$ and not only at t_{end} as in the last chapter with blind auctions. We have according to (3)

$$P^* = A_{t_{end}} = \left(S_{t_{end}} + \frac{z}{2}\right) + \lambda(X_0 - X_{t_{end}}) + D_{t_{end}}. \tag{44}$$

Since we use the same price determination as in the paper of Obizhaeva and Wang [20], one would think that the optimal execution strategy stays the same. But obviously the crucial difference is that the price for all bids during the auction $[t_{start}, t_{end}]$ is not determined until the end of the calling phase. Bidding during the auction increase the price for a discrete bid at t_{start}. Without auctions, earlier trades influenced the prices of later trades and not vice versa. For this reason, we have to reconsider the optimal execution strategy in the case with the described visible auctions. Questions that we address in this chapter are:
Are there bids on (t_{start}, t_{end})? Where do we put discrete bids and trades? Do we still buy evenly during the continuous trading phases?

7.1 Preparations

7.1.1 No order placement during auctions

In Chapter 6, other market participants did not react on our actions during the auction. Consequently, we restricted our strategies to one bid per auction. But for visible auctions, we have to ask ourselves how to allocate our bids during the calling phase. To answer this question, we will show in the following simple lemma that it is optimal in our setting to put all orders of the auction at the beginning of the calling phase as a single block. There is no order placement during the rest of the time.
We only consider the auction itself. Suppose that its bidding times $t_n = t_{start} + n\tau$ are fixed, where $\tau = \frac{t_{end} - t_{start}}{N}$ and $n = 0, ..., N$. As usual, we say $x_n \geq 0$ is the number of shares to be placed at time t_n.

Lemma 25. *If we want to buy a fixed number of shares $x \in \mathbb{R}_{>0}$ during an auction, then we will minimise our cost by choosing $x_0 = x$ and $x_n = 0$ for $n = 1, ..., N$.*

Proof: As indicated in (44), the auction price P^* is determined analogously to the case without auctions. That is

$$\begin{aligned}P^*(x_0,...,x_N) &= \left(S_{t_{end}}+\frac{z}{2}\right)+\lambda\left(X_0-X_{t_{start}}+\sum_{n=0}^{N}x_n\right)\\ &+ D_{t_{start}}e^{-\rho(t_{end}-t_{start})}+\kappa\sum_{n=0}^{N}x_n e^{-\rho(t_{end}-t_n)}.\end{aligned}$$

Consequently, we get the following optimisation problem

$$\min_{\left\{x_0,...,x_n\in\mathbb{R}\mid\sum_{n=0}^{N}x_n=x\right\}} P^*(x_0,...,x_N)\,x = \left(S_{t_{end}}+\frac{z}{2}\right)x+\lambda x\left(X_0-X_{t_{start}}+x\right)$$

$$+xD_{t_{start}}e^{-\rho(t_{end}-t_{start})}+\kappa x\min_{\left\{x_0,...,x_n\in\mathbb{R}\mid\sum_{n=0}^{N}x_n=x\right\}}\sum_{n=0}^{N}x_n e^{-\rho(t_{end}-t_n)}.$$

Obviously only the last term depends on $x_0,...,x_N$ and it is minimised by choosing $x_0=x$ and $x_n=0$ for $n=1,...,N$.

□

The lemma gives the following intuition: We have the same permanent impact in P^* no matter how we allocate the x shares during the auction, but the temporary impact in P^* will get smaller if we bid as early as possible.

7.1.2 Using time consistency to handle one auction

In this subsection we want to give a first idea of how to trade optimally if visible auctions in the above sense are involved. For this purpose, we consider the case where we start trading with an opening auction of length v_1 and there are no more auctions afterwards until T. Our aim is to show that it is optimal to put a discrete bid $\tilde{x}_0$ at the beginning of the auction, a discrete trade $\tilde{x}_{v_1}$ directly after the auction and a discrete trade $\tilde{x}_T$ at T. The remaining shares are purchased uniformly between the end of the auction and T. In order to prove this proposition, we need the lemma given below. It deals with the concept of time consistency, which we have already introduced in Remark 18.

Definition 26. An optimal trading strategy $(X_t)_{t\in[0,T]}$ is called **time-consistent** if recomputing the static trajectory in an arbitrary $\hat{t}\in(0,T)$ leads to the same trajectory, where the recomputation takes into account the temporary price impact $D_{\hat{t}}$ already accumulated and no new data is introduced.

In the following lemma, we show that the optimal strategy given in Proposition 13 and 15 for risk aversion $a=0$ is time-consistent. The lemma itself has nothing to do with auctions.

Lemma 27. *The optimal strategy $X_t=\frac{\rho(T-t)+1}{\kappa}D_t$ from (37) with constant temporary impact $D_t\equiv\kappa\frac{X_0}{\rho T+2}$ for $t\in(0,T)$ is time-consistent in terms of Definition 26.*

Proof: The proof is basically a recap of what was said in Remark 18.
Assume being at time $\hat{t}$ such that we have already accumulated a temporary price impact of $\tilde{D}$, and suppose there are $\tilde{X}$ shares still to be purchased until T. We now wonder how to choose the discrete trade $\tilde{x}_{\hat{t}}$ in $\hat{t}$ optimally. Because of $X_t = \frac{\rho(T-t)+1}{\kappa} D_t$ we get

$$\begin{aligned} \tilde{X} - \tilde{x}_{\hat{t}} &= X_{\hat{t}+} = \frac{\rho(T-\hat{t})+1}{\kappa} D_{\hat{t}+} = \frac{\rho(T-\hat{t})+1}{\kappa} \left(\tilde{D} + \kappa \tilde{x}_{\hat{t}} \right) \\ \Leftrightarrow \tilde{x}_{\hat{t}} &= \frac{1}{\rho(T-\hat{t})+2} \left(\tilde{X} - \frac{\rho(T-\hat{t})+1}{\kappa} \tilde{D} \right). \end{aligned} \tag{45}$$

If we act optimally until time $\hat{t}$, we will get

$$\tilde{X} = X_0 - x_0 - \hat{t} \frac{\rho X_0}{\rho T + 2} = X_0 \frac{\rho(T-\hat{t})+1}{\rho T + 2}$$

and $\tilde{D} = \kappa \frac{X_0}{\rho T + 2}$. Fitting this into (45) gives $\tilde{x}_{\hat{t}} = 0$. Thus, the optimal strategy is time-consistent. □

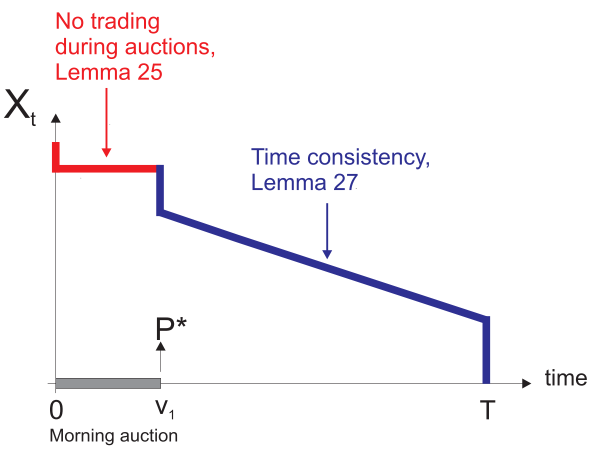


Figure 15: Optimal trajectory for one opening auction.

We can now apply Lemma 25 and 27 to obtain the result for one auction, which we already stated at the beginning of the subsection:
Lemma 25 tells us to put a discrete trade, say $\tilde{x}_0$, at the beginning of the morning auction and there is no more bidding during the auction. Then we can use (45) for $\hat{t} = v_1$ to see that we should optimally buy

$$\tilde{x}_{v_1} = \frac{1}{\rho(T-v_1)+2} \left(X_0 - \tilde{x}_0 - [\rho(T-v_1)+1] \tilde{x}_0 e^{-\rho v_1} \right)$$

shares at v_1. Because of $dD_t = 0$ for all $t \in (v_1, T]$, we get a constant $D_t = D_{v_1+} = \kappa x$ and $\tilde{\mu}_t = \frac{\rho}{\kappa} D_t = \rho x$ for all $t \in (v_1, T]$, where x is defined as

$$x := \tilde{x}_0 e^{-\rho v_1} + \tilde{x}_{v_1}.$$

Finally there are

$$\tilde{x}_T = X_T = \frac{1}{\kappa} D_T = x$$

shares left to be purchased at T as a discrete trade. A short test shows that

$$\tilde{x}_0 + \tilde{x}_{v_1} + (T - v_1)\tilde{\mu}_t + \tilde{x}_T$$

add up to X_0 as desired. Notice that we have not determined the size of $\tilde{x}_0$.
In the following we want to examine what happens if more than one auction is involved. An interesting question will be if we still trade uniformly between the auctions such as in Chapter 6.

7.2 Optimal strategy by backward induction

Applying auxiliary Lemma 25, we get the proposition below giving the optimal execution strategy in the case with several auctions when we consider discrete trading times. Again the proof can be done by backward induction involving dynamic programming techniques.
Suppose that we are given a purchase order whose execution should take $d \in \mathbb{N}$ days and let us assume that we have $N+1$ trading times per day. That is, we look again at the matrix of trading times $(t_{i,j})_{\substack{i=0,\ldots,N \\ j=1,\ldots,d}}$ and the associated matrix of discrete trades

$$(x_{i,j})_{\substack{i=0,\ldots,N \\ j=1,\ldots,d}}.$$

Since Lemma 25 tells us to place orders only at the beginning of an auction, we want to interpret $x_{0,j}$ and $x_{N,j}$ as the number of shares to be bid at the beginning of the j-th day's opening and closing auction. According to this $x_{1,j}$ and $x_{N-1,j}$ are the first and the last trade during the continuous trading phase of the j-th day. This leads to the following time partition

$$t_{1,j} - t_{0,j} = v_1,\ t_{0,j} - t_{N,j-1} = v_2 \text{ and } t_{i,j} - t_{i-1,j} = \tfrac{1}{N-1} \text{ for } i = 2, ..., N,$$

which is slightly different from (42) since we are bidding at the beginning instead of the end of the auctions.

We do not want to restrain trading to start at the beginning of the first day and to stop at the end of the last day. The investor can choose the parameters

$$first \text{ and } last \in \{0, ..., N\}$$

in advance of the execution. They represent the first trade on the first day and the last trade on the last day, i.e. $x_{i,1} = 0$ for $i = 0, ..., first-1$ and $x_{i,d} = 0$ for $i = last+1, ..., N$ as illustrated in Table 5. Thus, the time for buying the X_0 shares is given by $T = t_{last,d} - t_{first,1}$.
We denote by $P^*_{opening,j}$ and $P^*_{closing,j}$ the auction price of the opening and closing auction of the j-th day and $a'_{i,j}$ is again the successor of the matrix entry $a_{i,j}$.

Proposition 28. *(Optimal trading strategy with visible auctions)*
The optimal strategy that minimises the expected cost is

$$x_{i,j} = \frac{1}{2}\delta'_{i,j}\left[\epsilon'_{i,j}X_{i,j} - \phi'_{i,j}D_{i,j}\right]$$

$$\begin{array}{ll} j = 1 & i = first, ..., N \\ j = 2, .., d-1 & i = 0, ..., N \\ j = d & i = 0, ..., last - 1 \end{array}$$

and $x_{last,d} = X_{last,d}$*. The expected cost for future trades under the optimal strategy is*

$$C_{i,j} = \left(S_{i,j} + \frac{z}{2}\right)X_{i,j} + \lambda X_0 X_{i,j} + \left[\alpha_{i,j}X_{i,j}^2 + \beta_{i,j}X_{i,j}D_{i,j} + \gamma_{i,j}D_{i,j}^2\right],$$

where the coefficients α, β, γ, δ, ϵ *and* ϕ *are determined as follows:*

Initialisation:

$$\alpha_{last,d} = \begin{cases} \kappa e^{-\rho v_1} & \text{if } last = 0 \\ \kappa e^{-\rho v_2} & \text{if } last = N \\ \frac{1}{2q} - \lambda & \text{otherwise} \end{cases}, \quad \beta_{last,d} = \begin{cases} e^{-\rho v_1} & \text{if } last = 0 \\ e^{-\rho v_2} & \text{if } last = N \\ 1 & \text{otherwise} \end{cases}, \quad \gamma_{last,d} = 0$$

Backward recursion: *(indices* i, j *left out for convenience)*

$$\alpha = \alpha' - \tfrac{1}{4}\delta'(\epsilon')^2, \qquad \beta = \beta' e^{-\rho(t'-t)} + \tfrac{1}{2}\delta'\epsilon'\phi', \qquad \gamma = \gamma' e^{-2\rho(t'-t)} - \tfrac{1}{4}\delta'(\phi')^2$$

$$\delta' = \begin{cases} \left[(\kappa e^{-\rho(t'-t)} + \lambda) + \alpha' - \kappa\beta' e^{-\rho(t'-t)} + \kappa^2\gamma' e^{-2\rho(t'-t)}\right]^{-1} & \text{if } \delta' = \delta_{0,j} \text{ or } \delta_{1,j} \\ \left[\frac{1}{2q} + \alpha' - \kappa\beta' e^{-\rho(t'-t)} + \kappa^2\gamma' e^{-2\rho(t'-t)}\right]^{-1} & \text{otherwise} \end{cases}$$

$$\epsilon' = \lambda + 2\alpha' - \kappa\beta' e^{-\rho(t'-t)}$$

$$\phi' = \begin{cases} e^{-\rho(t'-t)} - \beta' e^{-\rho(t'-t)} + \kappa\gamma' e^{-2\rho(t'-t)} & \text{if } \phi' = \phi_{0,j} \text{ or } \phi_{1,j} \\ 1 - \beta' e^{-\rho(t'-t)} + \kappa\gamma' e^{-2\rho(t'-t)} & \text{otherwise} \end{cases}$$

Proof: We proof the structure of C by backward induction analogously to Lemma 4. That is we run backwards through the columns of the considered matrices. Firstly, we deal with $C_{last,d}$ by using case differentiation. In the case $last = 0$, we have with (44)

$$\begin{aligned} C_{last,d} &= X_{last,d}\mathbb{E}\left[P^*_{opening,d}|\mathcal{F}_{t_{last,d}}\right] \\ &= X_{last,d}\left[(S_{last,d} + \frac{z}{2}) + \lambda X_0 + (D_{last,d} + \kappa X_{last,d})\,e^{-\rho v_1}\right] \\ &= \left(S_{last,d} + \frac{z}{2}\right)X_{last,d} + \lambda X_0 X_{last,d} + \left[\kappa e^{-\rho v_1}X_{last,d}^2 + e^{-\rho v_1}X_{last,d}D_{last,d}\right]. \end{aligned}$$

We get the same term but with v_2 instead of v_1 in the case $last = N$.
For $last = 1, ..., N-1$ the induction basis is analogue to Lemma 4:

$$\begin{aligned} C_{last,d} &= P_{last,d} X_{last,d} \\ &= \left(S_{last,d} + \frac{z}{2}\right) X_{last,d} + \lambda X_0 X_{last,d} + \left[\left(\frac{1}{2q} - \lambda\right) X_{last,d}^2 + X_{last,d} D_{last,d}\right]. \end{aligned}$$

The inductive step can be seen by a case differentiation, too. We distinguish between the steps involving no auction and involving the opening or the closing auction. In the case where no auction is involved, we consider $C_{i,j}$ for an arbitrary j and $i \in \{1, ..., N-1\}$. The dynamics are exactly the same as in the inductive step of Lemma 4. Therefore, we only need to examine the dynamics of the morning and evening auctions, where we have $i = 0$ and $i = N$, respectively. We start with the morning auction:

$$C_{0,j} = \min_{x_{0,j} \in \mathbb{R}} \left\{ x_{0,j} \mathbb{E}\left[P^*_{opening,j} | \mathcal{F}_{t_{0,j}}\right] + \mathbb{E}\left[C_{1,j} | \mathcal{F}_{t_{0,j}}\right] \right\}. \tag{46}$$

We can now put

$$\mathbb{E}\left[P^*_{opening,j} | \mathcal{F}_{t_{0,j}}\right] = \left(S_{0,j} + \frac{z}{2}\right) + \lambda\left[(X_0 - X_{0,j}) + x_{0,j}\right] + (D_{0,j} + \kappa x_{0,j}) e^{-\rho v_1}$$

into (46) and apply the induction hypothesis to $C_{1,j}$ where

$$X_{1,j} = (X_{0,j} - x_{0,j}) \text{ and } D_{1,j} = (D_{0,j} + \kappa x_{0,j}) e^{-\rho v_1}.$$

The term we receive is quadratic in $x_{0,j}$ just as in the inductive step without auctions. Differentiating with respect to $x_{0,j}$ gives a minimum in $x_{0,j} = \frac{1}{2}\delta_{1,j}[\epsilon_{1,j} X_{0,j} - \phi_{1,j} D_{0,j}]$ where $\delta_{1,j}$, $\epsilon_{1,j}$ and $\phi_{1,j}$ are defined as stated in the recursion of Proposition 28.
Now we paste this $x_{0,j}$ into the term to be minimised, and after some calculation we obtain the desired result

$$C_{0,j} = \left(S_{0,j} + \frac{z}{2}\right) X_{0,j} + \lambda X_0 X_{0,j} + \left[\alpha_{0,j} X_{0,j}^2 + \beta_{0,j} X_{0,j} D_{0,j} + \gamma_{0,j} D_{0,j}^2\right].$$

In the case of the evening auction we have

$$C_{N,j} = \min_{x_{N,j} \in \mathbb{R}} \left\{ x_{N,j} \mathbb{E}\left[P^*_{closing,j} | \mathcal{F}_{t_{N,j}}\right] + \mathbb{E}\left[C_{0,j+1} | \mathcal{F}_{t_{N,j}}\right] \right\}.$$

instead of (46), and everything goes analogously to the morning auction with v_2 instead of v_1. □

7.3 Interpretation of the optimal strategy

By implementing a small program with adequate software, the optimal $x_{i,j}$ out of Proposition 28 can be calculated explicitly. Table 5 shows exemplary outputs of the program. From the first row of the table, you can see that the number of shares to be bid at the beginning of the morning auction $x_{0,j}$ is equal for $j = 2, ..., d$. The same is

	Output 1: $first = 6$, $last = 4$				Output 2: $first = 0$, $last = 10$				Output 3: $first = 4$, $last = 10$			
	1	2	3	4	1	2	3	4	1	2	3	4
x_0	0	0.96	0.96	0.96	1.22	0.75	0.75	0.75	0	0.80	0.80	0.80
x_1	0	2.99	2.99	2.99	8.51	2.33	2.33	2.33	0	2.49	2.49	2.49
x_2	0	2.72	2.72	2.72	2.11	2.11	2.11	2.11	0	2.26	2.26	2.26
x_3	0	2.72	2.72	2.72	2.11	2.11	2.11	2.11	0	2.26	2.26	2.26
x_4	0	2.72	2.72	12.28	2.11	2.11	2.11	2.11	10.22	2.26	2.26	2.26
x_5	0	2.72	2.72	0	2.11	2.11	2.11	2.11	2.26	2.26	2.26	2.26
x_6	12.28	2.72	2.72	0	2.11	2.11	2.11	2.11	2.26	2.26	2.26	2.26
x_7	2.72	2.72	2.72	0	2.11	2.11	2.11	2.11	2.26	2.26	2.26	2.26
x_8	2.72	2.72	2.72	0	2.11	2.11	2.11	2.11	2.26	2.26	2.26	2.26
x_9	3.81	3.81	3.81	0	2.96	2.96	2.96	7.90	3.17	3.17	3.17	8.45
x_{10}	1.10	1.10	1.10	0	0.85	0.85	0.85	2.49	0.913	0.913	0.913	2.66

Table 5: Optimal strategy in discrete time with visible auctions in the model of Obizhaeva and Wang for different choices of $first$ and $last$. The initial order to trade is set to $X_0 = 100{,}000$ shares, we consider $d = 4$ days and the number of discrete trades per day is $N = 10$, the market depth is $q = 5{,}000$ units, the permanent price impact parameter is $\lambda = \frac{1}{2q}$ and the resiliency coefficient is $\rho = 2$. The morning and the evening auction have the same length with $v_1 = v_2 = \frac{0.5}{6.5}$ meaning that an auction is modelled as 30 minutes in comparison to the 6.5 hours of continuous trading during one day. **The $x_{i,j}$ in the table are given in thousand shares.**

true for the discrete trades $x_{1,j}$ directly after the morning auction. We will call these values x_m and x'_m. The bids $x_{N,j}$ for the evening auctions and the trades $x_{N-1,j}$ directly prior to the evening auction, which we will call x_e and x'_e respectively, are identical for $j = 1, ..., d-1$.
If the auctions are modelled to have the same length, $v_1 = v_2$, then one will observe that the auction bids and the trades flanking the auctions become identical for large N, i.e.

$$\lim_{N\to\infty} x_m - x_e = \lim_{N\to\infty} x'_m - x'_e = 0.$$

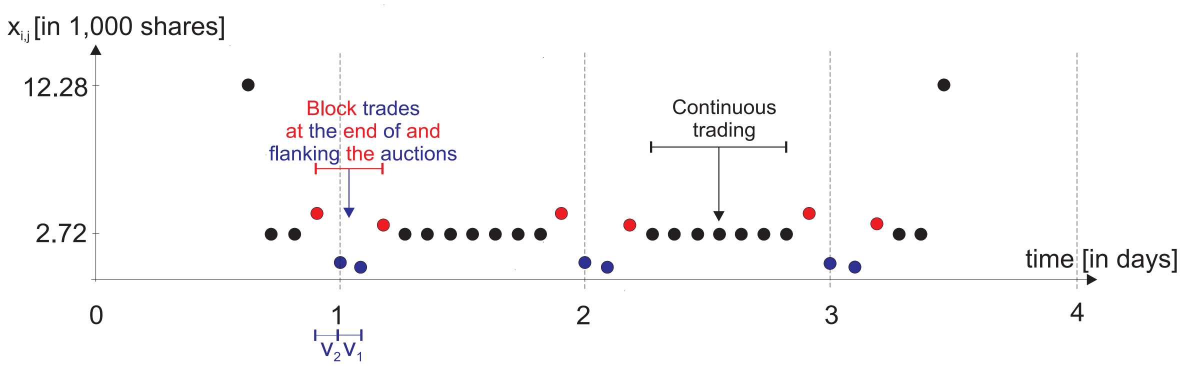


Figure 16: Schematic description of the optimal strategy "Output 1" from Table 5.

	$first = 0$	$first = 1, .., N-2$	$first = N-1$	$first = N$
$last = N$	x_{first} x'_{first} x_{last} x'_{last}	x_{first} x_{last} x'_{last}	x_{first} x_{last} x'_{last}	x_{first} x_{last} x'_{last}
$last = N-1, .., 2$	x_{first} x'_{first} x_{last}	$x_{first} (= x_{last})$	x_{first} x_{last}	$x_{first}, .., x''_{first}$ x_{last}
$last = 1$	x_{first} x'_{first} $x_{last}, .., x'''_{last}$	x_{first} $x_{last}, .., x'''_{last}$	x_{first} $x_{last}, .., x'''_{last}$	x_{first} $x_{last}, .., x'''_{last}$
$last = 0$	x_{first} x'_{first} $x_{last}, .., x''_{last}$	x_{first} $x_{last}, .., x''_{last}$	x_{first} $x_{last}, .., x''_{last}$	x_{first} $x_{last}, .., x''_{last}$

Table 6: The table shows how many different parameters occur in the optimal strategy in addition to x_m, x'_m, x_e and x'_e depending on the choice of $first$ and $last$. Thereby $x_{first} := x_{first,1}$, $x_{last} := x_{last,d}$ and x'_{first}, x'_{last} are the successor and the predecessor respectively of x_{first} and x_{last}. E.g. for $last = 0$ we have $x''_{last} = x_{N-1,d-1}$.

What happens at the beginning of trading on the first day and the end of trading on the last day, depends on the choice of the parameters $first$ and $last \in \{0, ..., N\}$. All possible cases are shown in Table 6. Here are some examples:
For $first \in \{1, ..., N-2\}$ and $last \in \{2, ..., N-1\}$ we have $x_{first,1} = x_{last,d}$ (see output 1). For $v_1 = v_2$ and $first = 0$ and $last = N$ (we trade during the entire first and last day), $x_{0,1}$ and $x_{N,d}$ as well as $x_{1,1}$ and $x_{N-1,d}$ converge to the same value for large numbers of N (see output 2).
Besides, it is remarkable that the trades $x_{i,j}$ for $i = 2, ..., N-2$ during continuous trading are equal. Tests showed that they converge to zero for large N. Moreover, the optimal strategy is linear in X_0. E.g. by doubling X_0 all $x_{i,j}$ are doubled as well.

The insights, described above, bring us to the following proposition when taking continuous instead of discrete trading time:
We get minimal cost to buy the X_0 shares by minimising over maximal 11 variables, namely x_{first},..., x''_{first}, x_{last},..., x'''_{last} to be found in Table 6 as well as x_m, x'_m, x_e, x'_e and by trading with a constant rate between the auctions. In the common case that $v_1 = v_2$ and we start trading at the beginning or the middle of the first day and stop in the middle or the end of the last day, we get three up to at most five variables to be optimised.

In this chapter we succeeded in modelling visible auctions in the Obizaheva and Wang context. The resulting optimal strategy is characterised by constant trading during the continuous trading phases and by discrete trades not only at the beginning of the auctions, but also next to them.

8 Closer inspection of the auction mechanism

So far we approached modelling auctions in a relatively basic manner: We adjusted our time partition by the volume times v_1 and v_2, argued that there is only one bid per auction and assumed similar price behaviour as in the Almgren, Chriss [2] and Obizhaeva, Wang [20] papers. Let us now analyse the composition of the auction price in more detail in order to understand what differing price impacts our bids on an auction might have.

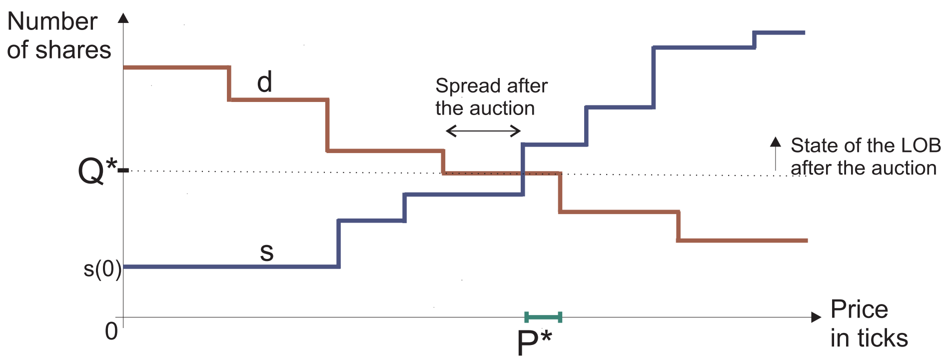


Figure 17: Accumulated orders in the LOB at the end of an auction.

At the end of an auction's calling phase, all limit and market orders in the LOB can be summarised in a diagram like Figure 17, which we will call **cumulative supply/demand curve**. Marked in red is the cumulative demand curve $d : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$. Hence, $d(p)$ is the total number of shares demanded for a price smaller than or equal to p. Therefore, d is a decreasing and left-continuous function. In an analogous manner, the blue supply curve is increasing and right-continuous. Both curves are step functions where a step in price p represents one or more limit orders over $d(p) - d(p+)$ shares in case of a buy and $s(p) - s(p-)$ shares in case of a sell order. Thereby $s(0)$ is the total number of shares resulting from market sell orders.
For a given price $p \in \mathbb{R}_{\geq}$,

$$\min(d(p), s(p))$$

is the number of shares that would be traded at this price. As mentioned before, the auction price P^* is the price which maximises the traded volume. Thus, P^* is the intersection of the demand and the supply curve[4]. If there are several prices P_i^* which lead to the maximum traded volume, the one with the smallest **imbalance** or sometimes called surplus $|\Delta Q_i^*|$ will be chosen[5], where

$$\Delta Q_i^* := d(P_i^*) - s(P_i^*).$$

[4]There will be no trading on the auction at all if the demand and supply curves are not overlapping. The spread is not crossed.

[5]If this rule still does not lead to a price decision, there will be successive rules, which depend on the marked place, but this goes beyond our scope. Let us just shortly pick a rule from the Frankfurt Stock Exchange as an example: If the imbalances all have same sign, the highest price P_i^* will be chosen in case of a positive sign, and vice versa for a negative sign–see the green price interval in Figure 17.

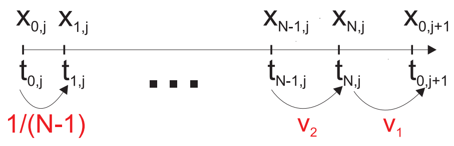
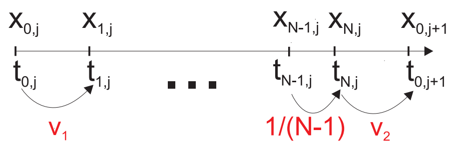
<table>
<tr><th></th><th>a) Blind</th><th>b) Visible</th></tr>
<tr><td>Used time partition:</td><td>Bidding at the auction end.
</td><td>Bidding at the auction beginning.
</td></tr>
<tr><td>Model I</td><td colspan="2">Higher resiliency during the auctions by using volume time v_1, v_2.</td></tr>
<tr><td></td><td>Subsection 6.2</td><td>Section 7</td></tr>
<tr><td>Model II</td><td colspan="2">Straight line with spread.
Higher market depth q_1, q_2 on the auctions.
Subsections 8.1 and 8.2</td></tr>
<tr><td>Model III</td><td colspan="2">Straight line with crossed spread.
Higher market depth q_1, q_2 on the auctions.
Subsections 8.1 and 8.2</td></tr>
</table>

Table 7: Overview of the call auction models.

If the imbalance of the chosen auction price is unequal to zero, a price-time priority rule will be used to decide which orders are executed. Hence, there is at most one order that is only partly executed.

Orders corresponding to the two branches of d and s below

$$Q^* := \min(d(P^*), s(P^*))$$

are traded on the auction and the two branches above this level are kept in the LOB and are forwarded to the following continuous trading phase or call auction respectively. The appropriate spread in the LOB after the auction is marked in the figure.

In [17], Mendelson analyses e.g. the expectation and variance of P^* and Q^*. He undertakes simplified assumptions: For example, he only considers limit orders with bounded uniformly distributed prices over one share. This means that the step functions have unit jumps.
To get a first intuition of the price impact on auctions, we should bear in mind that placing a market buy order comprising x_0 shares lifts the whole demand curve d up and therefore, increases the auction price P^*.

We will proceed with modelling the step functions as straight lines in the next subsections. This of course is equivalent to assuming a block form of the LOB as illustrated in Figure 2. Furthermore, we will try to incorporate the fact that there is higher liquidity during auctions, e.g. due to orders with the additional specification "auction only". In order to keep track of the models already introduced and the ones to be described in the next subsections, an overview is provided in Table 7.

8.1 Straight line: Explanation of Model II and III

We will start this subsection by explaining our Model II. As already hinted in Table 7, we take the supply and demand curves as the positive part of straight lines. The supply curve has slope q and is zero for prices between 0 and the best ask. The demand curve has slope $-q$ and is zero for prices larger than the best bid. This corresponds to the assumption that the same number of shares from market buy and sell orders is put on the auction by other market participants. In Model II, we assume that the best bid stays smaller than the best ask during the auction. This means that the supply and demand curves do not cross each other.

If we now put a market buy order of x_0 shares on the auction, P^* is the best ask price plus $\frac{x_0}{q}$. The argumentation why there is only one bid on an auction is analogous to Model I. However, the price is quite high in comparison to the price during continuous trading where we only have to pay an extra $\frac{x_n}{2q}$ on top of the best ask price at time t_n. But this high price is not intuitive, since the liquidity is usually higher on the auctions. Therefore, it seems sensible to assume higher market depths

$$q_1, q_2 > q$$

on the morning and evening auction, respectively. Thereby q denotes the market depth during continuous trading. We assume the ratio of the permanent impact to the total impact $\widehat{\lambda}$ to be constant and invent the process of the permanent price impact D^p analogous to Subsection 5.1. The dynamics of the average price per share and the permanent and temporary price impact can be found in Table 8. We define

$$\lambda_i := \frac{\widehat{\lambda}}{q_i} \quad \text{and} \quad \kappa_i := \frac{\widehat{\kappa}}{q_i} \quad \text{for } i = 1, 2.$$

Accordingly we set $\lambda := \frac{\widehat{\lambda}}{q}$ and $\kappa := \frac{\widehat{\kappa}}{q}$. Once again, a backward induction yields an optimal strategy to be found in the Appendix A.3. We included the already discussed Model I in these tables as well for a better comparison.

The interpretation of the optimal strategy for Model II is postponed to Subsection 8.2. We will firstly introduce another extension of our model.

<table>
<tr><th></th><th>a) Blind</th><th>b) Visible</th></tr>
<tr><td rowspan="2">I</td><td>$D_{0,j+1} = (D_{N,j} + \kappa x_{N,j})e^{-\rho v_1}$
$D_{N,j} = (D_{N-1,j} + \kappa x_{N-1,j})e^{-\rho v_2}$
$D_{n,j} = (D_{n-1,j} + \kappa x_{n-1,j})e^{-\rho \tau}$</td><td>$D_{1,j} = (D_{0,j} + \kappa x_{0,j})e^{-\rho v_1}$
$D_{0,j+1} = (D_{N,j} + \kappa x_{N,j})e^{-\rho v_2}$
$D_{n,j} = (D_{n-1,j} + \kappa x_{n-1,j})e^{-\rho \tau}$</td></tr>
<tr><td>$\overline{P}_{n,j} = S_{n,j} + \frac{z}{2} + \lambda(X_0 - X_{n,j}) + D_{n,j} + \frac{x_{n,j}}{2q}$</td><td>$\overline{P}_{0,j} = S_{1,j} + \frac{z}{2} + \lambda(X_0 - X_{0,j} + x_{0,j}) + D_{1,j}$
$\overline{P}_{N,j} = S_{0,j+1} + \frac{z}{2} + \lambda(X_0 - X_{N,j} + x_{N,j}) + D_{0,j+1}$
$\overline{P}_{n,j} = S_{n,j} + \frac{z}{2} + \lambda(X_0 - X_{n,j}) + D_{n,j} + \frac{x_{n,j}}{2q}$</td></tr>
<tr><td rowspan="5">II</td><td colspan="2">$D^p_{1,j} = D^p_{0,j} + \lambda_1 x_{0,j}$
$D^p_{0,j+1} = D^p_{N,j} + \lambda_2 x_{N,j}$
$D^p_{n,j} = D^p_{n-1,j} + \lambda x_{n-1,j}$</td></tr>
<tr><td>$D^t_{1,j} = (D^t_{0,j} + \kappa_1 x_{0,j})e^{-\rho \tau}$
$D^t_{0,j+1} = (D^t_{N,j} + \kappa_2 x_{N,j})e^{-\rho v_1}$
$D^t_{N,j} = (D^t_{N-1,j} + \kappa x_{N-1,j})e^{-\rho v_2}$
$D^t_{n,j} = (D^t_{n-1,j} + \kappa x_{n-1,j})e^{-\rho \tau}$</td><td>$D^t_{1,j} = (D^t_{0,j} + \kappa_1 x_{0,j})e^{-\rho v_1}$
$D^t_{0,j+1} = (D^t_{N,j} + \kappa_2 x_{N,j})e^{-\rho v_2}$
$D^t_{n,j} = (D^t_{n-1,j} + \kappa x_{n-1,j})e^{-\rho \tau}$</td></tr>
<tr><td>$\overline{P}_{0,j} = S_{0,j} + \frac{z}{2} + D^p_{0,j} + D^t_{0,j} + \frac{x_{0,j}}{q_1}$
$\overline{P}_{N,j} = S_{N,j} + \frac{z}{2} + D^p_{N,j} + D^t_{N,j} + \frac{x_{N,j}}{q_2}$</td><td>$\overline{P}_{0,j} = S_{1,j} + \frac{z}{2} + D^p_{1,j} + D^t_{1,j}$
$\overline{P}_{N,j} = S_{0,j+1} + \frac{z}{2} + D^p_{0,j+1} + D^t_{0,j+1}$</td></tr>
<tr><td colspan="2">$\overline{P}_{n,j} = S_{n,j} + \frac{z}{2} + D^p_{n,j} + D^t_{n,j} + \frac{x_{n,j}}{2q}$</td></tr>
<tr><td colspan="2"></td></tr>
<tr><td rowspan="5">III</td><td colspan="2">D^p: see II</td></tr>
<tr><td>$D^A_{1,j} = D^B_{1,j} = (D^A_{0,j} + \frac{1}{2}(\kappa_1 - \lambda_1)x_{0,j})e^{-\rho \tau}$
$D^A_{0,j+1} = D^B_{0,j+1} = (D^A_{N,j} + \frac{1}{2}(\kappa_2 - \lambda_2)x_{N,j})e^{-\rho v_1}$
$D^A_{N,j} = D^B_{N,j} = \frac{1}{2}(D^A_{N-1,j} + D^B_{N-1,j} + (\kappa - \lambda)x_{N-1,j})e^{-\rho v_2}$</td><td>$D^A_{1,j} = D^B_{1,j} = (D^A_{0,j} + \frac{1}{2}(\kappa_1 - \lambda_1)x_{0,j})e^{-\rho v_1}$
$D^A_{0,j+1} = D^B_{0,j+1} = (D^A_{N,j} + \frac{1}{2}(\kappa_2 - \lambda_2)x_{N,j})e^{-\rho v_2}$
$D^A_{N,j} = D^B_{N,j} = \frac{1}{2}(D^A_{N-1,j} + D^B_{N-1,j} + (\kappa - \lambda)x_{N-1,j})e^{-\rho \tau}$</td></tr>
<tr><td colspan="2">$D^A_{n,j} = (D^A_{n-1,j} + \kappa x_{n-1,j})e^{-\rho \tau}$
$D^B_{n,j} = (D^B_{n-1,j} - \lambda x_{n-1,j})e^{-\rho \tau}$</td></tr>
<tr><td>$\overline{P}_{0,j} = S_{0,j} + \frac{z}{2} + D^p_{0,j} + D^A_{0,j} + \frac{x_{0,j}}{2q_1}$
$\overline{P}_{N,j} = S_{N,j} + \frac{z}{2} + D^p_{N,j} + D^A_{N,j} + \frac{x_{N,j}}{2q_2}$</td><td>$\overline{P}_{0,j} = S_{1,j} + \frac{z}{2} + D^p_{1,j} + D^A_{1,j}$
$\overline{P}_{N,j} = S_{0,j+1} + \frac{z}{2} + D^p_{0,j+1} + D^A_{0,j+1}$</td></tr>
<tr><td colspan="2">$\overline{P}_{n,j} = S_{n,j} + \frac{z}{2} + D^p_{n,j} + D^A_{n,j} + \frac{x_{n,j}}{2q}$</td></tr>
</table>

Table 8: Dynamics of the impact processes and average prices per share for the various models.

The idea of Model III is to allow for a crossed spread during the auctions. This means that our auction price will not only depend on the best ask price A at the end of the auction, but also on the best bid price B. Indeed we have not modelled the developing of the best bid price, i.e. the left hand side of our LOB, in dependence on our trading strategy so far. For simplicity and without loss of generality we will neglect the constant spread z in our following considerations.

Then our model of the best ask price is as before (see (3))

$$A_t = S_t + \lambda\left(X_0 - X_t\right) + \kappa \sum_{i=1}^{n(t)} x_i e^{-\rho_A(t-t_i)}, \tag{47}$$

where $n(t) = \max_{\{t_i<t\}} i$. Our best bid price should also be affected by the permanent price impact. But reasonably this should not happen instantly. This fact motivates the following model for the best bid price

$$B_t = S_t + \lambda\left(X_0 - X_t\right) - \lambda \sum_{i=1}^{n(t)} x_i e^{-\rho_B(t-t_i)}. \tag{48}$$

When we stop trading and let t go to infinity, both A_t and B_t converge to

$$\widehat{S}_t = S_t + \lambda(X_0 - X_t).$$

That is, ρ_A and ρ_B are the resiliency speeds of this convergence. We now want to address the question of what happens during an auction with starting and end time t_{start} and t_{end} on which we trade x_0 shares.
Let us assume that the best ask and bid price evolve on $[t_{start}, t_{end}]$ as given in (47) and (48). In this way, it is guaranteed that $B_t \leq A_t$. This is absolutely fine during continuous trading, but we usually experience a crossed spread during auctions due to high liquidity. Therefore we assume that A and B move towards each other by an appropriately large price $c > 0$ on each auction. "Appropriately large" means that we want to be able to compute our auction price as

$$P^* = \frac{A_{t_{end}} + B_{t_{end}}}{2} + \frac{x_0}{2q}.$$

This implies that we have to assume c to satisfy the following inequality which is visualised through the green gradient triangle in Figure 18:

$$A_{t_{end}} - c \leq B_{t_{end}} + c - \frac{x_0}{q}.$$

With the above assumption our auction price per share thus is

$$\begin{aligned} P^* &= \frac{A_{t_{end}} + B_{t_{end}}}{2} + \frac{x_0}{2q} \\ &= S_{t_{end}} + \lambda\left(X_0 - X_{t_{end}}\right) + \frac{1}{2}\left[\kappa \sum_{i=1}^{n(t_{end})} x_i e^{-\rho_A(t_{end}-t_i)} - \lambda \sum_{i=1}^{n(t_{end})} x_i e^{-\rho_B(t_{end}-t_i)} + \frac{x_0}{q}\right]. \end{aligned}$$

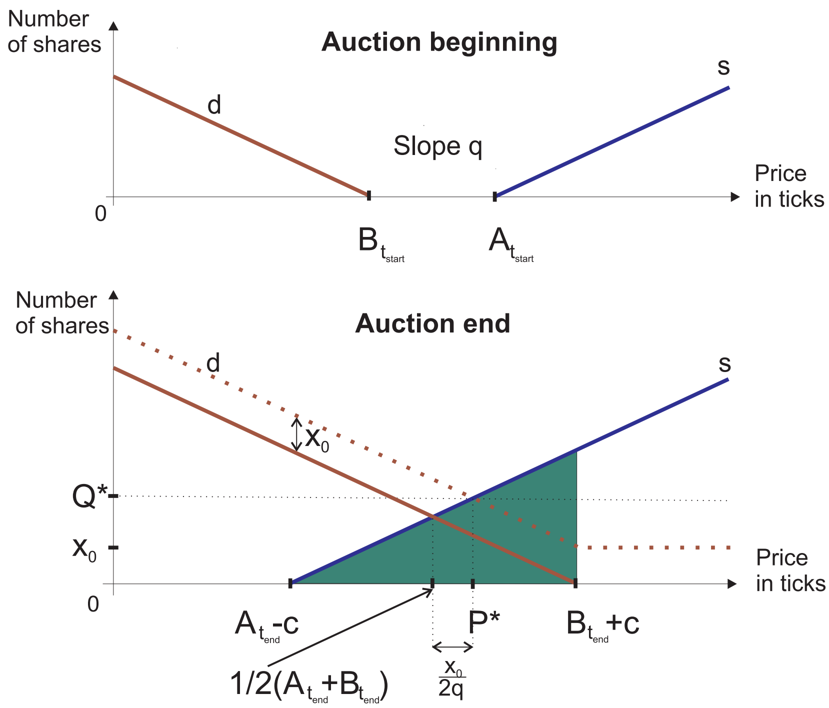


Figure 18: Schematic picture of the LOB during an auction to illustrate Model III.

For the best bid and ask prices after the auction we have

$$A_{t_{end}+} = B_{t_{end}+} = P^*.$$

After all, we can summarise

$$\begin{aligned} A_t &= S_t + D_t^p + D_t^A \quad \text{and} \\ B_t &= S_t + D_t^p + D_t^B \quad \text{for } t \in [0, T] \end{aligned} \tag{49}$$

for the permanent impact D^p and the temporary impact D^A and D^B as given in Table 8 under III. In the table different market depths q_1 and q_2, as explained above, are additionally inserted into the processes and the average price per share for blind auctions finally is

$$\begin{aligned} \overline{P}_{0,j} &= A_{0,j} + \frac{x_{0,j}}{2q_1}, \\ \overline{P}_{N,j} &= A_{N,j} + \frac{x_{N,j}}{2q_2} \quad \text{and} \\ \overline{P}_{n,j} &= A_{n,j} + \frac{x_{n,j}}{2q} \quad \text{for } n = 1, .., N-1. \end{aligned}$$

It is worth mentioning that we had to set $\rho_A = \rho_B$ (see (47) and (48)) to be able to formulate the best ask and bid prices as in (49) and to do our backward induction. The results are given in Appendix A.3.

8.2 Straight line: Optimal strategies

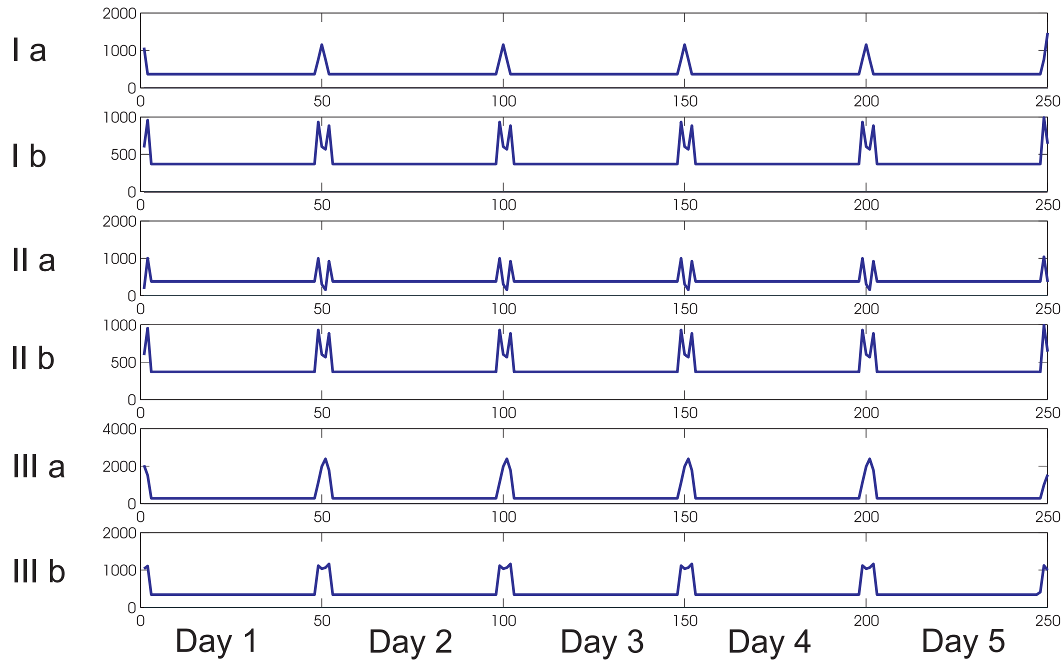


Figure 19: Optimal strategies $(x_{n,j})$ for the various models as given in Table 7 for $q_1 = q_2 = q$. We have chosen $X_0 = 100{,}000$ and $q = 5{,}000$ shares, $\rho = 20$, $\widehat{\lambda} = \frac{2}{3}$, $v_1 = v_2 = \frac{0.5}{6.5}$ and considered a period of $d = 5$ days and $N = 49$ trading times per day.

In Figure 19 you see explicit optimal strategies as given in Table A.3 for the various models in the special case that $\mathbf{q_1} = \mathbf{q_2} = \mathbf{q}$. Please note that the axes of ordinates have different scaling. The peaks mark the auction phases. That is

$$x_{N-1,j},\ x_{N,j},\ x_{0,j+1} \text{ and } x_{1,j+1} \text{ for } j = 1, ..., d-1$$

are of utmost importance. Each of the optimal strategies for the various models have the same two characteristics:

- The level of trading during continuous trading is constant not only over one day but also over all days $j = 1, ..., d$.
- The auction phases are identical. This means that for every fixed $i = N-1, N, 0$ or 1, $x_{i,j} = x_{i,j+1}$ for all $j = 1, ..., d-1$.

The appearance of the auction phases depends heavily on the model and the chosen parameters.

Let us now turn to the case where the market depth on the morning and evening auction q_1 and q_2 are bigger than q. As Figure 20 suggests, the two characteristics mentioned above are not true anymore for the strategies of the Models II and III where q_1 and q_2 are relevant: Indeed the trading level during continuous trading is

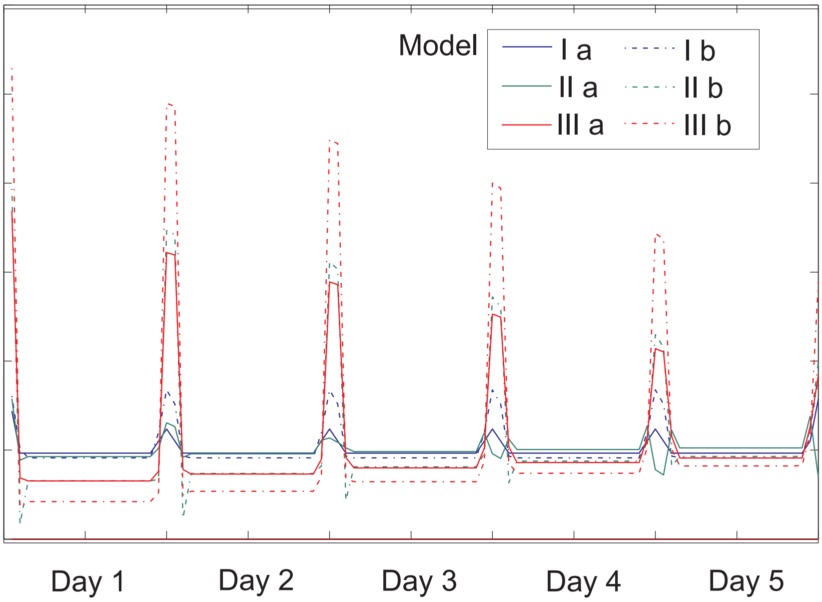


Figure 20: Optimal strategies $(x_{n,j})$ for the various models for $q_1 = q_2 = 1.05q$. We have chosen $X_0 = 100{,}000$ and $q = 5{,}000$ shares, $\rho = 20$, $\widehat{\lambda} = \frac{1}{3}$, $v_1 = v_2 = \frac{0.5}{6.5}$ and considered a period of $d = 5$ days and $N = 19$ trading times per day.

constant during one day, but this constant changes daily and the auction profiles are not identical anymore. Moreover, the parameters for the optimisation have to be chosen very carefully. That is q_1 and $q_2 > q$ have to be small enough. Otherwise we get negative values for some $x_{n,j}$. The strategies become unusual, since we only modelled buying and not selling of shares.

Getting a closer insight into the auction price determination gave the idea to model the supply and demand curves on auctions as straight lines. The optimal strategies that we derived are indeed similar to the ones from Chapter 7, but their auction phases are different. It ultimately depends on the market place which of the presented models are applicable.

9 Generalisation of the block shape of the LOB

9.1 Exponential decay of the area

Inspired by the paper of Obizhaeva and Wang [20], we have considered a block shape of the LOB so far. Now we want to analyse more general forms of the LOB. To simplify matters, we do not take into account permanent impact. The form of the LOB is described by a continuous, positive function $f : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$. The best ask and bid price at time zero are modelled as before as

$$A_0 = S_0 + \frac{z}{2} \qquad \text{and} \qquad B_0 = S_0 - \frac{z}{2},$$

where S is the Bachelier model and z the constant spread. Since we are only buying shares, we neglect the left hand side of the LOB. An illustration can be found in Figure 21.

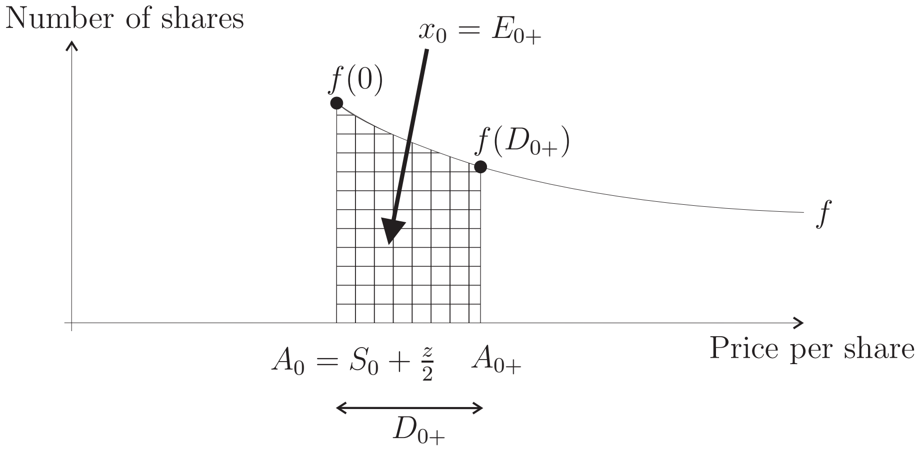


Figure 21: Illustration of the processes D and E.

We now describe how a purchase of x_0 shares at time zero effects the LOB and how much we pay for this purchase. The best ask will be temporarily increased by an **extra spread** D_{0+} to the level

$$A_{0+} = A_0 + D_{0+}.$$

Let F be the antiderivative of the LOB form f. Then this extra spread D_{0+} is defined via

$$\int_0^{D_{0+}} f(x)dx = x_0 \quad \Leftrightarrow \quad D_{0+} = F^{-1}(x_0).$$

The cost we have to pay for the total purchase of the x_0 shares is

$$(S_0 + \frac{z}{2})x_0 + \int_0^{D_{0+}} xf(x)dx. \tag{50}$$

This means that we receive the total cost by multiplying the price per share (x) and the amount ($f(x)$). The price per share lies in the whole range $[S_0 + \frac{z}{2}, S_0 + \frac{z}{2} + D_{0+}]$– the cheapest shares within the package of the x_0 shares are executed at A_0 per share, but we are charged up to A_{0+} for the more expensive ones in the package. If we divide (50) by x_0, we will get the average price per share. This motivates the following definition of the average price per share at time t_n:

$$\overline{P}_{t_n} = S_{t_n} + \frac{z}{2} + \frac{1}{x_n} \int_{D_{t_n-}}^{D_{t_n+}} x f(x) dx, \tag{51}$$

where we trade at equidistant discrete trading times $t_n = n\tau$ for $n = 0, ..., N$ and $\tau := \frac{T}{N}$. Thereby x_n is the number of shares we trade at t_n.
To complete our model, we still need to specify the dynamic of the extra spread D_t or the resiliency effect, respectively. We do that be defining the process E_t of the shares already eaten up from the LOB:

$$E_t = \int_0^{D_t} f(x) dx. \tag{52}$$

It is simply the area that corresponds to the extra spread (see Figure 21) and (52) is equivalent to

$$D_t = F^{-1}(E_t). \tag{53}$$

Both D and E are only relevant at the discrete trading times t_n. Using (52) and (53), we can transfer one process into the other.
As in the Obizhaeva and Wang paper [20], we assume that the resiliency of the LOB leads to an exponential decrease– the question is if E_t or D_t is exponentially decreased, which only makes a difference if the LOB form f is not constant. Both alternatives seem reasonable and therefore we want to examine both of them. The exponential decrease of the extra spread D is considered in Subsection 9.2. We start by dealing with the area decaying model and set:

$$E_{0-} = 0, \; E_{t_n+} = E_{t_n-} + x_n, \; E_{t_{k+1}-} = e^{-\rho\tau} E_{t_k+} = e^{-\rho\tau} \left(E_{t_k-} + x_k \right) \tag{54}$$

for $n = 0, ..., N$ and $k = 0, ..., N-1$. As in the Obizhaeva and Wang model [20], the constant ρ is the resiliency speed of the LOB. Incidentally, it would be possible to understand the processes E and D to be left-continuous, but instead we use the notation D_{t_n-} and D_{t_n+}, which turns out to be more convenient.

Let us consider the antiderivative F of the LOB shape f and the integral which emerges in the price term (51):

$$F(x) := \int_0^x f(y) dy \quad \text{and} \quad \widetilde{F}(x) := \int_0^x y f(y) dy.$$

Then F, F^{-1} and $\widetilde{F}$ are continuously differentiable and strictly increasing on $\mathbb{R}$.
The function F is assumed to be unbounded in the sense that

$$\lim_{x\to\infty} F(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} F(x) = -\infty.$$

As before, we want to buy X_0 shares until time T and minimise the expectation of the cost. To this end we look for an optimal strategy in the set of all static strategies

$$\Xi := \{(x_0, ..., x_N) \in \mathbb{R}^{N+1} | \sum_{n=0}^{N} x_n = X_0\}.$$

The expected cost incurred by such a strategy is

$$C_0\left(x_0, ..., x_N\right) := \mathbb{E}\left[\sum_{n=0}^{N} x_n \overline{P}_{t_n}\right].$$

Consequently, our optimisation problem is

$$C_0 := \min_{\Xi} \mathbb{E}\left[\sum_{n=0}^{N} x_n \overline{P}_{t_n}\right]. \tag{55}$$

Proposition 29. *(Optimal strategy for exponentially decaying area)*
Suppose that the function

$$h_E(u) := F^{-1}(u) - e^{-\rho\tau} F^{-1}(e^{-\rho\tau} u)$$

is one-to-one. Then the optimal strategy is

$$x_1 = ... = x_{N-1} = x_0\left(1 - e^{-\rho\tau}\right), \quad x_N = X_0 - x_0 - (N-1)x_0\left(1 - e^{-\rho\tau}\right) \tag{56}$$

and the initial trade x_0 is uniquely defined by the equation

$$F^{-1}\left(X_0 - N x_0\left(1 - e^{-\rho\tau}\right)\right) = \frac{h_E(x_0)}{1 - e^{-\rho\tau}}. \tag{57}$$

In particular, this implies $x_0, ..., x_N > 0$ and

$$E_{t_n -} = x_0 e^{-\rho\tau} \quad \textit{for } n = 1, ..., N. \tag{58}$$

Remark 30. According to the dynamic (54), (56) and (58), the left hand side of (57) is equal to D_{t_N+}:

$$\begin{aligned}
D_{t_N+} &= F^{-1}\left(E_{t_N+}\right) = F^{-1}\left(E_{t_N-} + x_N\right) \\
&= F^{-1}\left(x_0 e^{-\rho\tau} + X_0 - x_0 - (N-1)x_0\left(1 - e^{-\rho\tau}\right)\right) \\
&= F^{-1}\left(X_0 - N x_0\left(1 - e^{-\rho\tau}\right)\right).
\end{aligned}$$

That is the optimal strategy depends only on the initial trade x_0, which itself depends on f and can be computed by equation (57). Besides, the proposition tells us that $x_1, ..., x_{N-1}$ are of equal size just as in the Obizhaeva and Wang model [20], which assumes f being constant. Furthermore, the optimal strategy is characterised by the fact that the initial trade eats up part of the LOB and therefore shifts the process E to the level $E_{0+} = x_0$. Afterwards the following trades $x_1, .., x_{N-1}$ consume exactly the limit sell orders that newly flow into the book, due to the resiliency of the LOB, such that the process E stays constant. At the end, at time t_N, the remaining shares are bought. A few examples can be found in Subsection 9.3.

Proof of Proposition 29: The cost functional can be written as

$$\begin{aligned}
C_0(x_0, ..., x_N) &= \left(S_0 + \frac{z}{2}\right) X_0 + \sum_{n=0}^{N} \int_{D_{t_n-}}^{D_{t_n+}} x f(x) dx \\
&= \left(S_0 + \frac{z}{2}\right) X_0 + \sum_{n=0}^{N} \left(\widetilde{F}\left(F^{-1}\left(E_{t_n+}\right)\right) - \widetilde{F}\left(F^{-1}\left(E_{t_n-}\right)\right)\right).
\end{aligned}$$

Hence, with

$$G(x) := \widetilde{F}\left(F^{-1}(x)\right)$$

our minimisation problem is equivalent to the minimisation of

$$\begin{aligned}
\widetilde{C}_0\left(x_0, ..., x_N\right) &= \sum_{n=0}^{N} \left(G\left(E_{t_n-} + x_n\right) - G\left(E_{t_n-}\right)\right) \qquad (59) \\
&= \quad G\left(x_0\right) - G\left(0\right) \\
&\quad +G\left(x_0 e^{-\rho\tau} + x_1\right) - G\left(x_0 e^{-\rho\tau}\right) \\
&\quad +G\left(x_0 e^{-2\rho\tau} + x_1 e^{-\rho\tau} + x_2\right) - G\left(x_0 e^{-2\rho\tau} + x_1 e^{-\rho\tau}\right) \\
&\quad +... \\
&\quad +G\left(x_0 e^{-N\rho\tau} + ... + x_N\right) - G\left(x_0 e^{-N\rho\tau} + ... + x_{N-1} e^{-\rho\tau}\right).
\end{aligned}$$

The derivative of G is

$$G'(x) = \widetilde{F}'\left(F^{-1}(x)\right)\left(F^{-1}\right)'(x) = F^{-1}(x) f\left(F^{-1}(x)\right) \frac{1}{f(F^{-1}(x))} = F^{-1}(x). \qquad (60)$$

Therefore, G is twice continuously differentiable, positive and convex. Furthermore G is increasing for positive x and decreasing for negative x. Therefore the following inequality holds for all $x \in \mathbb{R}$ and $c \in [0, 1]$

$$G(x) - G(cx) \geq (1 - c)|G'(x)||x| = (1 - c)|F^{-1}(x)||x|. \qquad (61)$$

It will be useful in Lemma 31.

After these preparations the proof proceeds as follows: In Step i) we prove that the optimal strategy has the form (56) and in ii) we derive equation (57). In the last step the uniqueness and the positivity of the strategy are shown.

Step i)
In order to have the existence of a Lagrange multiplier we need the following lemma.

Lemma 31. *There exists a local minimum of $\widetilde{C}_0$ in Ξ.*

Proof of Proposition 31: We prove this lemma by showing that there is a cost explosion for extreme trading strategies $\vec{x} = (x_0, ..., x_N) \in \mathbb{R}^{N+1}$, i.e. $\widetilde{C}_0(x_0, ..., x_N)$ converges to infinity for $\|\vec{x}\| \to \infty$. We start by rearranging the sum in (59) in order to use inequality (61). We obtain

$$\begin{aligned}
&\widetilde{C}_0(x_0,...,x_N)\\
&= G\left(x_0e^{-N\rho\tau}+...+x_N\right)-G\left(0\right)\\
&\quad+\sum_{n=0}^{N-1} G(e^{-n\rho\tau}x_0+...+x_n)-G(e^{-\rho\tau}(e^{-n\rho\tau}x_0+...+x_n))\\
&\geq G\left(x_0e^{-N\rho\tau}+...+x_N\right)-G\left(0\right)\\
&\quad+(1-e^{-\rho\tau})\sum_{n=0}^{N-1}\left|F^{-1}(e^{-\rho\tau}(e^{-n\rho\tau}x_0+...+x_n))\right|\left|e^{-n\rho\tau}x_0+...+x_n\right|.
\end{aligned}$$

Because of F being unbounded, we know that both $G(x)$ and $|F^{-1}(e^{-\rho\tau}x)||x|$ converge to infinity for $|x|\to\infty$. It is easy to see that at least one of the terms $|e^{-n\rho\tau}x_0+...+x_n|$ converge to infinity for $\|\vec{x}\|\to\infty$. This proves Lemma 31. □

Due to Lemma 31, we can apply Theorem 4, page 109 of [10] which you find in the Appendix A.4 in the form we are using it here. According to this, there exists a Lagrange multiplier $\nu\in\mathbb{R}$ such that the optimal strategy satisfies

$$\frac{\partial}{\partial x_j}\widetilde{C}_0(x_0,...,x_N)=\nu$$

for $j=0,...,N$. Now we use the form of $\widetilde{C}_0$ as given in (59) to obtain the following connection between the partial derivatives of $\widetilde{C}_0$ for $j=0,...,N-1$:

$$\begin{aligned}
\frac{\partial}{\partial x_j}\widetilde{C}_0(x_0,...,x_N) &= e^{-\rho\tau}\left[\frac{\partial}{\partial x_{j+1}}\widetilde{C}_0(x_0,...,x_N)-G'\left(e^{-\rho\tau}(x_0e^{-j\rho\tau}+...+x_j)\right)\right]\\
&+ G'\left(x_0e^{-j\rho\tau}+...+x_j\right)
\end{aligned}$$

Recalling (60), we can compute the optimal strategy from the equations

$$h_E\left(x_0e^{-j\rho\tau}+...+x_j\right)=\nu\left(1-e^{-\rho\tau}\right)\qquad\text{for } j=0,...,N-1.$$

Hence, we get the following optimal strategy

$$\begin{aligned}
x_0 &= h_E^{-1}\left(\nu\left(1-e^{-\rho\tau}\right)\right)\\
x_j &= x_0\left(1-e^{-\rho\tau}\right)\qquad\text{for } j=0,...,N-1\\
x_N &= X_0-x_0-(N-1)x_0\left(1-e^{-\rho\tau}\right).
\end{aligned}\tag{62}$$

In this situation we need h_E to be one-to-one.
Note that the uniqueness of the optimal strategy will be shown in Step iii).

Step ii)
We would now like to know in more detail how to choose the optimal initial trade x_0 and therefore consider the term

$$\begin{aligned}
\widetilde{C}_0(x_0) &:= \widetilde{C}_0(x_0,x_0\left(1-e^{-\rho\tau}\right),...,x_0\left(1-e^{-\rho\tau}\right),X_0-x_0-(N-1)x_0\left(1-e^{-\rho\tau}\right))\\
&= G(x_0)-G(0)+(N-1)\left[G\left(x_0e^{-\rho\tau}+x_0\left(1-e^{-\rho\tau}\right)\right)-G\left(x_0e^{-\rho\tau}\right)\right]+\\
&\quad G\left(x_0e^{-\rho\tau}+X_0-x_0-(N-1)x_0\left(1-e^{-\rho\tau}\right)\right)-G\left(x_0e^{-\rho\tau}\right)\\
&= N\left[G(x_0)-G\left(x_0e^{-\rho\tau}\right)\right]+G\left(X_0+Nx_0\left(e^{-\rho\tau}-1\right)\right)-G(0)
\end{aligned}\tag{63}$$

in order to minimise with respect to x_0. We have used the fact that the optimal strategy (62) satisfies

$$E_{t_n-} = e^{-\rho\tau}\left(x_0 e^{-\rho\tau} + x_0\left(1-e^{-\rho\tau}\right)\right) = x_0 e^{-\rho\tau}$$

due to the dynamics (54). Differentiating (63) with respect to x_0 and resolving for the minimum gives

$$\begin{aligned} &\frac{d\widetilde{C}_0(x_0)}{dx_0} = \\ &\quad N\left[G'(x_0) - e^{-\rho\tau}G'\left(x_0 e^{-\rho\tau}\right) + \left(e^{-\rho\tau}-1\right)G'\left(X_0 + Nx_0\left(e^{-\rho\tau}-1\right)\right)\right] \end{aligned} \tag{64}$$

Equaling this to zero implies

$$\begin{aligned} &e^{-\rho\tau}\left[G'\left(X_0 + Nx_0\left(e^{-\rho\tau}-1\right)\right) - G'\left(x_0 e^{-\rho\tau}\right)\right] \\ &= G'\left(X_0 + Nx_0\left(e^{-\rho\tau}-1\right)\right) - G'(x_0). \end{aligned} \tag{65}$$

This will also be needed later on. Because of $G'(x) = F^{-1}(x)$, (65) implies that $\widetilde{C}_0(x_0)$ will become minimal if the initial trade x_0 satisfies the condition

$$\left(1-e^{-\rho\tau}\right)F^{-1}\left(X_0 + Nx_0\left(e^{-\rho\tau}-1\right)\right) = F^{-1}(x_0) - e^{-\rho\tau}F^{-1}(x_0 e^{-\rho\tau}).$$

Step iii)
We conclude by proving the uniqueness of the optimal strategy, which according to the preceding steps is equivalent to the uniqueness of the Lagrange multiplier ν. Recalling (64), $\frac{\partial}{\partial x_0}\widetilde{C}_0(x_0) = N\hat{h}_E(x_0)$ with

$$\hat{h}_E(u) := h_E(u) - \left(1-e^{-\rho\tau}\right)F^{-1}(X_0 - Nu\left(1-e^{-\rho\tau}\right)).$$

Thus we will be finished if we can show that $\hat{h}_E$ is strictly increasing and therefore has at most one zero.
We know that $h_E(0) = 0$, $h_E(u) > 0$ for $u > 0$ and h_E is continuous and one-to-one. Consequently h_E is strictly increasing and therefore

$$\hat{h}'_E(u) = h'_E(u) + \frac{N\left(e^{-\rho\tau}-1\right)^2}{f\left(F^{-1}\left(X_0 + Nu\left(e^{-\rho\tau}-1\right)\right)\right)} > 0.$$

Furthermore, the positivity of the optimal initial trade x_0 follows immediately, since

$$\hat{h}_E(0) = \left(e^{-\rho\tau}-1\right)F^{-1}(X_0) < 0.$$

The last trade x_N is strictly positive as well, since assuming $x_N \leq 0$ would mean that $D_{t_N+} \leq D_{t_1-} < D_{0+}$ which is a contradiction to $e^{-\rho\tau}\left[D_{t_N+} - D_{t_1-}\right] = D_{t_N+} - D_{0+}$ as given in (65). $\square$

Remark 32. (When is h_E one-to-one?)
As mentioned in the proof of Proposition 29, h_E is continuous with $h_E(0) = 0$ and $h_E(u) > 0$ for $u > 0$. That means that h_E is one-to-one if and only if h_E is strictly increasing. We want to consider when this is the case:

$$h'_E(u) = \frac{1}{f(F^{-1}(u))} - \frac{e^{-2\rho\tau}}{f(F^{-1}(e^{-\rho\tau}u))} > 0$$

for all $u \in \mathbb{R}$ is equivalent to

$$l(u) := f(F^{-1}(e^{-\rho\tau}u)) - e^{-2\rho\tau} f(F^{-1}(u)) > 0. \tag{66}$$

That is, the function h_E will be one-to-one for instance if the assumed LOB shape function f is decreasing for $u > 0$ and increasing for $u < 0$.

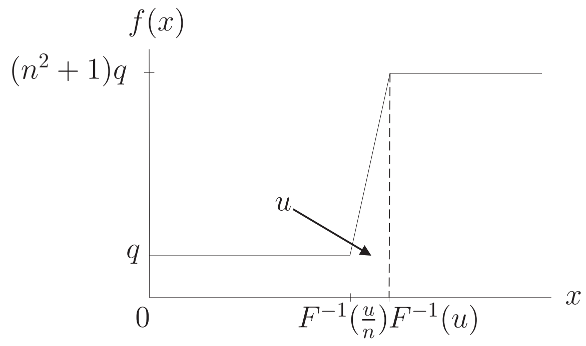


Figure 22: The figure shows an example for a function f such that the corresponding function h_E is not one-to-one.

We now want to give an example for a LOB shape f such that the corresponding h_E is not one-to-one. For this purpose, we assume that there exists an $n \in \{2, 3, ...\}$ such that $e^{-\rho\tau} = \frac{1}{n}$. We set as plotted in Figure 22

$$f(x) := \begin{cases} q & x \in [0, \frac{\frac{1}{2}n^2+1}{n-1}) \\ q + n^2 q(x - \frac{\frac{1}{2}n^2+1}{n-1}) & x \in \left[\frac{\frac{1}{2}n^2+1}{n-1}, \frac{\frac{1}{2}n^2+1}{n-1} + 1\right] \\ (n^2+1)q & x \in \left(\frac{\frac{1}{2}n^2+1}{n-1} + 1, \infty\right) \end{cases}$$

and $u := (\frac{\frac{1}{2}n^2+1}{n-1} + 1)q + \frac{1}{2}n^2 q$ such that $F^{-1}(u) = \frac{\frac{1}{2}n^2+1}{n-1} + 1$ as well as

$$F^{-1}(e^{-\rho\tau}u) = F^{-1}(\frac{u}{n}) = \frac{\frac{1}{2}n^2+1}{n-1}.$$

Hence, we get

$$f\left(F^{-1}\left(e^{-\rho\tau}u\right)\right) = q < q\left(1 + \frac{1}{n^2}\right) = e^{-2\rho\tau} f\left(F^{-1}(u)\right),$$

which tells us according to (66) that the corresponding h_E is not strictly increasing.

9.2 Exponential decay of the extra spread

We now want to change the model introduced in the last section in the following way. Instead of the exponential decay of the process E of the shares already eaten up, we want D to decay exponentially. Of course this does not make a difference if f is constant (see Example 0 in Subsection 9.3). That is D has the dynamics

$$D_{0-} = 0, \quad D_{t_n+} = F^{-1}\left(x_n + F\left(D_{t_n-}\right)\right) \quad \text{and} \quad D_{t_{n+1}-} = e^{-\rho\tau} D_{t_n+}. \tag{67}$$

The second equation in (67) is motivated by

$$\int_{D_{t_n-}}^{D_{t_n+}} f(x)dx = x_n.$$

All other definitions such as the price process (51) and our optimisation problem (55) remain as in Subsection 9.1.

Proposition 33. *(Optimal strategy for exponentially decaying extra spread) Suppose that the function*

$$h_D(x) := x\frac{f(x) - e^{-2\rho\tau} f(e^{-\rho\tau}x)}{f(x) - e^{-\rho\tau} f(e^{-\rho\tau}x)}$$

is one-to-one and that $\widetilde{F}$ is convex. Then the optimal strategy is

$$\begin{aligned} x_1 &= \ldots = x_{N-1} = x_0 - F\left(e^{-\rho\tau}F^{-1}(x_0)\right), \\ x_N &= X_0 - Nx_0 + (N-1)F\left(e^{-\rho\tau}F^{-1}(x_0)\right) \end{aligned} \tag{68}$$

and the initial trade x_0 is uniquely defined by the equation

$$F^{-1}\left(X_0 - N[x_0 - F(e^{-\rho\tau}F^{-1}(x_0))]\right) = h_D\left(F^{-1}(x_0)\right). \tag{69}$$

In particular, this implies $x_0, ..., x_N > 0$ and $D_{t_n-} = e^{-\rho\tau}F^{-1}(x_0)$ for $n = 1, ..., N$.

Remark 34. Similar to Proposition 29, we can show that the left hand side of (69) is equal to D_{t_N+}. According to the dynamic (67) and (68) we have

$$D_{0+} = F^{-1}(x_0) \qquad D_{t_1-} = e^{-\rho\tau}D_{0+} \qquad D_{t_N+} = F^{-1}\left(X_0 - N[x_0 - F(D_{t_1-})]\right).$$

The proposition tells us that the initial trade x_0 eats a gap into the LOB which stays constant until t_N-. The trades x_1 to x_{N-1} are of same size and only consume the limit sell orders that newly flow into the book due to the resiliency effect. That is, we have an optimal level of the extra spread of $D_{t_n-} = e^{-\rho\tau}F^{-1}(x_0)$ in comparison to Proposition 29 where we had $E_{t_n-} = x_0e^{-\rho\tau}$. This corresponds to $D_{t_n-} = F^{-1}\left(e^{-\rho\tau}x_0\right)$.

Proof of Proposition 33: Again our cost are

$$C_0 = \left(S_0 + \frac{z}{2}\right)X_0 + \min_{\Xi}\widetilde{C}_0(x_0, ..., x_N),$$

but with

$$\widetilde{C}_0(x_0, ..., x_N) := \sum_{n=0}^{N} \left(G\left(x_n + F\left(D_{t_n-}\right)\right) - \widetilde{F}\left(D_{t_n-}\right) \right) \tag{70}$$

and $F, \widetilde{F}$ and G as given in Subsection 9.1. The structure of the proceeding proof is similar to Proposition 29: We start by showing that there exists a local minimum of $\widetilde{C}_0$ in Ξ. To do so, we need the convexity of $\widetilde{F}$. In i) we then derive the optimal strategy as given in (68). In ii) we then show that the optimal initial trade x_0 is characterised by (69). Finally, we prove the uniqueness and positivity of the optimal strategy in Step iii).

Lemma 35. *There exists a local minimum of $\widetilde{C}_0$ in Ξ.*

Proof of Lemma 35: As for Lemma 31, we show that there is a cost explosion for extreme trading strategies $\vec{x} = (x_0, ..., x_N) \in \mathbb{R}^{N+1}$. Rearranging the sum in (70) gives

$$\begin{aligned} \widetilde{C}_0(x_0, ..., x_N) &= \sum_{n=0}^{N-1} \left[\widetilde{F}\left(F^{-1}(x_n + F(D_{t_n-}))\right) - \widetilde{F}\left(e^{-\rho\tau} F^{-1}(x_n + F(D_{t_n-}))\right) \right] \\ &+ \widetilde{F}\left(F^{-1}(x_N + F(D_{t_N-}))\right). \end{aligned} \tag{71}$$

The function $\widetilde{F}$ is increasing for positive x, decreasing for negative x and we assumed it to be convex. Therefore the following inequality holds

$$\widetilde{F}(x) - \widetilde{F}(e^{-\rho\tau}x) \geq (1 - e^{-\rho\tau})|\widetilde{F}'(e^{-\rho\tau}x)||x| = (1 - e^{-\rho\tau})e^{-\rho\tau}x^2 f(e^{-\rho\tau}x).$$

It can be used in (71) to obtain

$$\begin{aligned} &\widetilde{C}_0(x_0, ..., x_N) \\ &\geq (1 - e^{-\rho\tau})e^{-\rho\tau} \sum_{n=0}^{N-1} \left(F^{-1}\left(x_n + F(D_{t_n-})\right)\right)^2 f\left(e^{-\rho\tau} F^{-1}(x_n + F(D_{t_n-}))\right) \\ &\quad + \widetilde{F}\left(F^{-1}(x_N + F(D_{t_N-}))\right). \end{aligned}$$

Because of F being unbounded, we know that both $(F^{-1}(x))^2$ and $\widetilde{F}(F^{-1}(x)) = G(x)$ converge to infinity for $|x| \to \infty$. Moreover, at least one of the terms $|x_n + F(D_{t_n-})|$ for $n = 0, .., N$ converge to infinity for $\|\vec{x}\| \to \infty$ because D_{t_n-} depends on $x_0, ..., x_{n-1}$ only. This proves Lemma 35. $\square$

Step i)
We use the form of $\widetilde{C}_0$ as given in (70) to receive the following connection between the partial derivatives of $\widetilde{C}_0$ for $i = 0, ..., N-1$ (see Lemma 36):

$$\begin{aligned} &\frac{\partial}{\partial x_i} \widetilde{C}_0(x_0, ..., x_N) = \\ &F^{-1}\left(x_i + F\left(D_{t_i-}\right)\right) + \frac{e^{-\rho\tau} f\left(D_{t_{i+1}-}\right)}{f\left(F^{-1}\left(x_i + F\left(D_{t_i-}\right)\right)\right)} \left[\frac{\partial}{\partial x_{i+1}} \widetilde{C}_0(x_0, ..., x_N) - D_{t_{i+1}-} \right] \end{aligned} \tag{72}$$

Analogously to the proof of Proposition 29, we can use Lemma 35 and the theorem given in Appendix A.4 to guarantee the existence of a Lagrange multiplier ν such that (72) is equivalent to

$$\nu = h_D\left(F^{-1}\left(x_i + F\left(D_{t_i-}\right)\right)\right)$$

for $i = 0, ..., N-1$. Hence, we get the following optimal strategy

$$\begin{aligned} x_0 &= F\left(h_D^{-1}(\nu)\right) \\ x_i &= x_0 - F(D_{t_i-}) \\ &= x_0 - F\left(e^{-\rho\tau}F^{-1}(x_0)\right) \quad \text{for } i = 1, ..., N-1 \\ x_N &= X_0 - x_0 - (N-1)\left[x_0 - F\left(e^{-\rho\tau}F^{-1}(x_0)\right)\right]. \end{aligned} \tag{73}$$

We applied that

$$D_{t_i-} = e^{-\rho\tau}F^{-1}(x_0) \tag{74}$$

for $i = 1, ..., N$ as one sees by using induction and (67):

$$D_{t_i-} = e^{-\rho\tau}F^{-1}\left(x_0 - F\left(e^{-\rho\tau}F^{-1}(x_0)\right) + F\left(e^{-\rho\tau}F^{-1}(x_0)\right)\right) = e^{-\rho\tau}F^{-1}(x_0).$$

Incidentally, we needed h_D^{-1} to be one-to-one in (73).

Step ii)

Now we would like to know in more detail how to choose the optimal initial trade x_0 and therefore consider the term

$$\begin{aligned} \widetilde{C}_0(x_0) &:= \widetilde{C}_0\left(x_0, x_0 - F\left(e^{-\rho\tau}F^{-1}(x_0)\right), ..., X_0 - Nx_0 + (N-1)F\left(e^{-\rho\tau}F^{-1}(x_0)\right)\right) \\ &= N\left[G(x_0) - \widetilde{F}\left(e^{-\rho\tau}F^{-1}(x_0)\right)\right] + G\left(X_0 + N\left[F\left(e^{-\rho\tau}F^{-1}(x_0)\right) - x_0\right]\right). \end{aligned}$$

We used again that $D_{t_i-} = e^{-\rho\tau}F^{-1}(x_0)$ for $i = 1, ..., N$ as explained in (74). Differentiating with respect to x_0 gives (69) as desired:

$$\begin{aligned} & \frac{\partial}{\partial x_0}\widetilde{C}_0(x_0) = N\left[D_{0+} - e^{-2\rho\tau}D_{0+}\frac{f(D_{t_1-})}{f(D_{0+})} + D_{t_N+}(e^{-\rho\tau}\frac{f(D_{t_1-})}{f(D_{0+})} - 1)\right] \stackrel{!}{=} 0 \\ \Leftrightarrow \quad & D_{t_N+} = D_{0+}\frac{f(D_{0+}) - e^{-2\rho\tau}f(D_{t_1-})}{f(D_{0+}) - e^{-\rho\tau}f(D_{t_1-})} \end{aligned}$$

Step iii)

We conclude the proof by showing the uniqueness and the positivity of the optimal strategy:

Similar to Proposition 29, the optimal strategy is unique because equation (69) has at most one solution x_0. We prove this by showing that

$$\hat{h}_D(x) := h_D\left(F^{-1}(x)\right) - F^{-1}\left(X_0 - N\left[x - F\left(e^{-\rho\tau}F^{-1}(x)\right)\right]\right)$$

is strictly increasing. We have

$$\hat{h}'_D(x) = \frac{h'_D\left(F^{-1}(x)\right)}{f\left(F^{-1}(x)\right)} + N\frac{f\left(F^{-1}(x)\right) - e^{-\rho\tau}f\left(e^{-\rho\tau}F^{-1}(x)\right)}{f\left(F^{-1}(x)\right)f\left(F^{-1}\left(X_0 - N\left[x - F\left(e^{-\rho\tau}F^{-1}(x)\right)\right]\right)\right)} > 0,$$

since h_D is increasing and the numerator of the second term

$$h_1(x) := f(x) - e^{-\rho\tau} f(e^{-\rho\tau} x)$$

is strictly positive for all $x \in \mathbb{R}$ due to h_D being one-to-one. We see this as follows: We also define

$$h_2(x) := f(x) - e^{-2\rho\tau} f(e^{-\rho\tau} x)$$

with $h_1(x) < h_2(x)$ for all $x \in \mathbb{R}$ and

$$h_D(x) = x \frac{h_2(x)}{h_1(x)}.$$

Besides, we can compute $h_D(0) = 0$ and $h'_D(0) = \frac{1-e^{-2\rho\tau}}{1-e^{-\rho\tau}} > 0$. Hence, h_D is strictly increasing and positive, since h_D is one-to-one. Therefore the denominator h_1 of h_D has to be strictly positive.

The positivity of x_0 is clear because $\hat{h}_D(0) = -F^{-1}(X_0) < 0$. Thus,

$$x_i = x_0 - F(e^{-\rho\tau} F^{-1}(x_0)) > 0$$

for $i = 1, ..., N-1$. So it only remains to show that $x_N > 0$. Let us thereto assume that $x_0, x_1 > 0$ but $x_N \leq 0$. That is $D_{0+} > 0$ and

$$D_{t_N+} = D_{0+} \frac{f(D_{0+}) - e^{-2\rho\tau} f(e^{-\rho\tau} D_{0+})}{f(D_{0+}) - e^{-\rho\tau} f(e^{-\rho\tau} D_{0+})} \leq D_{t_1-} = e^{-\rho\tau} D_{0+}.$$

This gives a contradiction because the denominator $f(D_{0+}) - e^{-\rho\tau} f(e^{-\rho\tau} D_{0+})$ is strictly positive as we already argued above.

□

Lemma 36. *(Partial derivatives of $\widetilde{C}_0$)*
We have the following recursive scheme for the derivatives of $\widetilde{C}_0(x_0, ..., x_N)$ for $i = 0, ..., N-1$:

$$\frac{\partial}{\partial x_i} \widetilde{C}_0(x_0, ..., x_N) = \tag{75}$$

$$F^{-1}\left(x_i + F(D_{t_i-})\right) + \frac{e^{-\rho\tau} f\left(D_{t_{i+1}-}\right)}{f\left(F^{-1}\left(x_i + F(D_{t_i-})\right)\right)} \left[\frac{\partial}{\partial x_{i+1}} \widetilde{C}_0(x_0, ..., x_N) - D_{t_{i+1}-}\right]$$

Proof: Using (67) one can compute that for a fixed $n \in 1, .., N$ we have

$$\frac{\partial}{\partial x_i} D_{t_n-}(x_0, ..., x_{n-1}) = \frac{e^{-\rho\tau} f(D_{t_{i+1}-})}{f\left(F^{-1}\left(x_i + F(D_{t_i-})\right)\right)} \frac{\partial}{\partial x_{i+1}} D_{t_n-}(x_0, ..., x_{n-1}) \tag{76}$$

for $i = 0, ..., n-2$. Furthermore one calculates that, according to (70) and (60),

$$\begin{aligned} \frac{\partial}{\partial x_i} \widetilde{C}_0(x_0, ..., x_N) &= F^{-1}\left(x_i + F(D_{t_i-})\right) \qquad (77) \\ &+ \sum_{n=i+1}^{N} f(D_{t_n-}) \frac{\partial}{\partial x_i} D_{t_n-}(x_0, ..., x_{n-1}) \left[F^{-1}\left(x_n + F(D_{t_n-})\right) - D_{t_n-}\right] \end{aligned}$$

for $i = 0, .., N$. Plugging both (77) and (76) into (75) gives a true statement. □

Remark 37. (When is h_D one-to-one?)
To complete this section we now give an example for a LOB form f such that the corresponding h_D is not one-to-one. Since h_D is continuous and $h_D(0) = 0$, we will be done if we can specify $x_-, x_+ > 0$ such that $h_D(x_-) < 0$ and $h_D(x_+) > 0$. For this purpose, we assume that there exist $n \in \{2, 3, ...\}$ such that $e^{-\rho\tau} = \frac{1}{n}$ and set f as plotted in Figure 23 to

$$f(x) := \begin{cases} (n+1)q & x \in \left[0, \frac{1}{n}\right) \\ (n+1)q - \frac{n^2 q}{n-1}\left(x - \frac{1}{n}\right) & x \in \left[\frac{1}{n}, 1\right] \\ q & x \in (1, \infty) \end{cases}$$

Furthermore, we define $x_- := 1$ and $x_+ := \frac{1}{n}$ to obtain

$$h_D(x_-) = \frac{n^2 - (n+1)}{-n} < 0 \quad \text{and} \quad h_D(x_+) = \frac{1 - e^{-2\rho\tau}}{n\left(1 - e^{-\rho\tau}\right)} > 0.$$

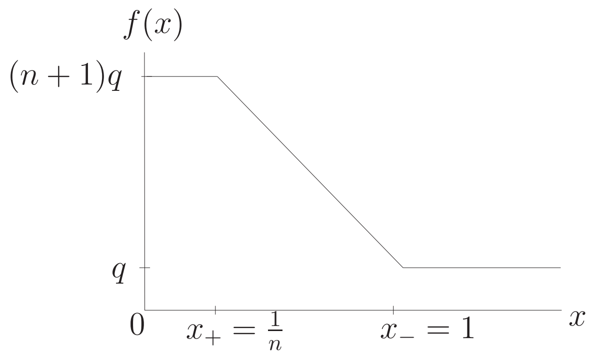


Figure 23: The figure shows an example for a function f such that the corresponding h_D is not one-to-one.

9.3 Examples

Let us refer to the area decaying model from Subsection 9.1 as model a) and let the one where the extra spread decays be model b). We first consider the block form $f(x) \equiv q > 0$ for the LOB shape f. This corresponds to the Obizhaeva and Wang model with permanent impact constant $\lambda = 0$. Then we have $F(x) = qx$, $F^{-1}(x) = \frac{x}{q}$ and

$$D_{0+} = \frac{x_0}{q} \qquad D_{t_1-} = e^{-\rho\tau}\frac{x_0}{q} \qquad D_{t_N+} = e^{-\rho\tau}\frac{x_0}{q} + \frac{X_0 - x_0 - (N-1)x_0\left(1 - e^{-\rho\tau}\right)}{q} \tag{78}$$

for model a) as well as b). Since the corresponding functions h_E and h_D are one-to-one for a constant $f \equiv q$, we can apply Proposition 29 and 33. Plugging (78) into (57) and (69) respectively and solving for x_0 gives in both cases

$$x_0 = \frac{X_0}{(N-1)\left(1 - e^{-\rho\tau}\right) + 2},$$

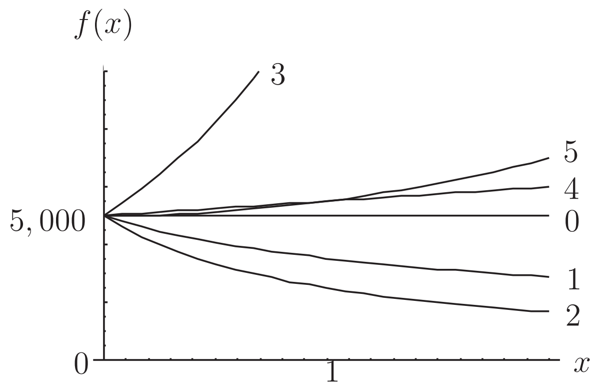


Figure 24: Exemplary LOB forms as given in Table 9 for $q = 5{,}000$ shares. The example numbers are given at the right hand side.

		Model a)			Model b)		
	$f(x)$	x_0	$x_1 = ... = x_{N-1}$	x_N	x_0	$x_1 = ... = x_{N-1}$	x_N
0	q	10,223	8,839	10,223	10,223	8,839	10,223
1	$\frac{q}{\sqrt{x+1}}$	10,257	8,869	9,925	10,756	8,724	10,726
2	$\frac{q}{x+1}$	10,303	8,909	9,520	13,305	8,154	13,305
3	qe^x	10,139	8,767	10,962	9,735	8,947	9,741
4	$\frac{q}{10}x + q$	10,211	8,829	10,326	10,130	8,860	10,131
5	$\frac{q}{10}x^2 + q$	10,192	8,812	10,498	10,101	8,868	10,091

Table 9: The table shows optimal strategies for various exemplary choices of the LOB form f. We set $X_0 = 100{,}000$ and $q = 5{,}000$ shares, $\rho = 20$, $T = 1$ and $N = 10$.

which is exactly what we got in Proposition 1 of Chapter 3.

We can now consider various other examples taking into account that

$$D_{0+} = F^{-1}(x_0) \qquad D_{t_1-} = F^{-1}\left(x_0 e^{-\rho\tau}\right) \qquad D_{t_N+} = F^{-1}\left(X_0 - N x_0\left(1 - e^{-\rho\tau}\right)\right)$$

in case of model a) and when considering model b) we have

$$D_{0+} = F^{-1}(x_0) \qquad D_{t_1-} = e^{-\rho\tau} D_{0+} \qquad D_{t_N+} = F^{-1}\left(X_0 - N\left[x_0 - F(D_{t_1-})\right]\right).$$

Plugging this into (57) and (69) respectively, we can explicitly derive the optimal initial trade x_0. By using (56) and (68), $x_1, ..., x_N$ can be computed. A few results can be found in Table 9. We check in the appendix that Proposition 29 and 33 can be applied tor these examples and therefore the existence of an optimal strategy is guaranteed. Incidentally, explicit calculations with the LOB shape $f(x) = \frac{q}{\sqrt{x+1}}$, having nice functions F, F^{-1} and $\widetilde{F}$, gave the idea to Proposition 29.

Recapitulatory, we were not satisfied with the block shape and therefore allowed a more general form f of the LOB. We wanted to know how this changes our optimal strategy in the risk-neutral and discrete trading time case without permanent impact. Astonishingly we still obtain two large trades at time 0 and T and constant trading in between regardless of whether the area or the extra spread is decaying.

A Appendix

A.1 Overview of the used constants and processes

The explicit parameter values are set as in the Obizhaeva and Wang paper to make the results comparable.

Constants with example values

z: Bid-ask spread ($z = 2$ ticks)
q: Market depth ($q = 5{,}000$ shares)
λ: Constant of the permanent price impact
($\lambda = \frac{1}{2q} = 10^{-4}$ ticks per share)
κ: Constant of the temporary price impact
($\kappa := \frac{1}{q} - \lambda = 10^{-4}$ ticks per share)
$\widehat{\lambda}$: Ratio of the permanent to the total price impact ($\widehat{\kappa} = 0.5$)
$\widehat{\kappa}$: Ratio of the temporary to the total price impact ($\widehat{\kappa} := 1 - \widehat{\lambda}$)
ρ: Resiliency of the LOB ($\rho = 2.31$)
ϑ: Half-life of the LOB ($\vartheta = 117$ minutes)
T: Given end of the trading horizon
($T = 1$ trading day= 6.5 trading hours)
N: $N + 1$ is the total number of trading times $t_0, ..., t_N$ in the discrete time case

Processes

A: Best ask price in the LOB
B: Best bid price in the LOB
C: Cost under the optimal strategy still to be paid
D: Temporary impact, i.e. deviation of the intrinsic best ask price ($\hat{S} + \frac{z}{2}$) and the actual best ask price A
$\overline{P}$: The average price per share achieved by the trader
S: Equilibrium asset price with volatility σ ($S_0 = 4{,}000$ ticks)
$\hat{S}$: S plus the permanent price impact
X: Number of shares left to be acquired ($X_0 = 100{,}000$ shares)

A.2 Sequences of Corollary 22

$$
\begin{array}{|llll|}
\hline
\alpha_N = \frac{1}{2q_N} & \text{and} & \alpha_n = & \alpha_{n+1} + \frac{1}{2}a\int_{t_n}^{t_{n+1}} \sigma_t^2 dt - \frac{1}{4}\delta_{n+1}\epsilon_{n+1}^2 \\
\beta_N = 1 & \text{and} & \beta_n = & \beta_{n+1}\exp\left(-\int_{t_n}^{t_{n+1}} \rho_t dt\right) + \frac{1}{2}\delta_{n+1}\epsilon_{n+1}\phi_{n+1} \\
\gamma_N = 0 & \text{and} & \gamma_n = & \gamma_{n+1}\exp\left(-2\int_{t_n}^{t_{n+1}} \rho_t dt\right) - \frac{1}{4}\delta_{n+1}\phi_{n+1}^2 \\
 & & & \\
\eta_N = \frac{z_N}{2} & \text{and} & \eta_n = & \eta_{n+1} + \frac{1}{2}\delta_{n+1}\epsilon_{n+1}\varphi_{n+1} \\
\mu_N = 0 & \text{and} & \mu_n = & \mu_{n+1}\exp\left(-\int_{t_n}^{t_{n+1}} \rho_t dt\right) - \frac{1}{2}\delta_{n+1}\phi_{n+1}\varphi_{n+1} \\
\omega_N = 0 & \text{and} & \omega_n = & \omega_{n+1} - \frac{1}{4}\delta_{n+1}\varphi_{n+1}^2 \\
 & & & \\
\multicolumn{4}{|l|}{\delta_n = \Big[\frac{1}{2q_{n-1}} - \frac{\widehat{\lambda}}{q_{n-1}} + \alpha_n + \frac{1}{2}a\int_{t_{n-1}}^{t_n} \sigma_t^2 dt} \\
\multicolumn{4}{|l|}{\qquad -\frac{\widehat{\kappa}}{q_{n-1}}\exp\left(-\int_{t_{n-1}}^{t_n} \rho_t dt\right)\beta_n + \left(\frac{\widehat{\kappa}}{q_{n-1}}\right)^2\exp\left(-2\int_{t_{n-1}}^{t_n} \rho_t dt\right)\gamma_n\Big]^{-1}} \\
\multicolumn{4}{|l|}{\epsilon_n = 2\alpha_n + a\int_{t_{n-1}}^{t_n} \sigma_t^2 dt - \frac{\widehat{\lambda}}{q_{n-1}} - \frac{\widehat{\kappa}}{q_{n-1}}\exp\left(-\int_{t_{n-1}}^{t_n} \rho_t dt\right)\beta_n} \\
\multicolumn{4}{|l|}{\phi_n = 1 - \exp\left(-\int_{t_{n-1}}^{t_n} \rho_t dt\right)\beta_n + 2\frac{\widehat{\kappa}}{q_{n-1}}\exp\left(-2\int_{t_{n-1}}^{t_n} \rho_t dt\right)\gamma_n} \\
\multicolumn{4}{|l|}{\varphi_n = \frac{z_{n-1}}{2} - \eta_n + \mu_n\frac{\widehat{\kappa}}{q_{n-1}}\exp\left(-\int_{t_{n-1}}^{t_n} \rho_t dt\right)} \\
\hline
\end{array}
$$

A.3 Results of the backward induction for the auction models I to III

In the following table, the results of the backward induction for the various auction models I to III are given in the order: cost structure, optimal strategy, initialisation and backward recursion of the used sequences.

<table>
<tr><th></th><th>a) Blind</th><th>b) Visible</th></tr>
<tr><td>I</td><td colspan="2">$C_{n,j} = (S_{n,j} + \frac{z}{2})X_{n,j} + \lambda X_0 X_{n,j} + \alpha_{n,j} X_{n,j}^2 + \beta_{n,j} X_{n,j} D_{n,j} + \gamma_{n,j} D_{n,j}^2$
$x_{n,j} = \frac{1}{2}\delta_{n+1,j}[\epsilon_{n+1,j} X_{n,j} - \phi_{n+1,j} D_{n,j}]$</td></tr>
<tr><td></td><td>$\alpha_{last,d} = \frac{1}{2q} - \lambda,\ \beta_{last,d} = 1$
$\tau_n^a = \{v_1 \text{ if } n = 0, v_2 \text{ if } n = N,\ \tau \text{ otherwise}\}$
$\alpha = \alpha' - \frac{1}{4}\delta'\epsilon'^2$
$\beta = \beta' e^{-\rho(\tau^a)'} + \frac{1}{2}\delta'\epsilon'\phi'$
$\gamma = \gamma' e^{-2\rho(\tau^a)'} - \frac{1}{4}\delta'\phi'^2$
$\delta = \left(\frac{1}{2q} + \alpha - \kappa e^{-\rho\tau^a}\beta + \kappa^2 e^{-2\rho\tau^a}\gamma\right)^{-1}$
$\epsilon = \lambda + 2\alpha - \kappa e^{-\rho\tau^a}\beta$
$\phi = 1 - e^{-\rho\tau^a}\beta + 2\kappa e^{-2\rho\tau^a}\gamma$</td><td>$\alpha_{last,d} = \begin{cases} \kappa e^{-\rho v_1} \\ \kappa e^{-\rho v_2} \\ \frac{1}{2q} - \lambda \end{cases},\quad \beta_{last,d} = \begin{cases} e^{-\rho v_1} & \text{if } last = 0 \\ e^{-\rho v_2} & \text{if } last = N \\ 1 & \text{otherwise} \end{cases}$
$\tau_n^b = \{v_1 \text{ if } n = 1, v_2 \text{ if } n = 0,\ \tau \text{ otherwise}\}$
$\alpha = \alpha' - \frac{1}{4}\delta'\epsilon'^2$
$\beta = \beta' e^{-\rho(\tau^b)'} + \frac{1}{2}\delta'\epsilon'\phi'$
$\gamma = \gamma' e^{-2\rho(\tau^b)'} - \frac{1}{4}\delta'\phi'^2$
$\delta = \left(\alpha - \kappa e^{-\rho\tau^b}\beta + \kappa^2 e^{-2\rho\tau^b}\gamma + \begin{cases} \kappa e^{-\rho v_1} + \lambda & \text{if } n = 1 \\ \kappa e^{-\rho v_2} + \lambda & \text{if } n = 0 \\ \frac{1}{2q} & \text{otherwise} \end{cases}\right)^{-1}$
$\epsilon = \lambda + 2\alpha - \kappa e^{-\rho\tau^b}\beta$
$\phi = -e^{-\rho\tau^b}\beta + 2\kappa e^{-2\rho\tau^b}\gamma + \begin{cases} e^{-\rho\tau^b} & \text{if } n = 0, 1 \\ 1 & \text{otherwise} \end{cases}$</td></tr>
</table>

II	$C_{n,j} = (S_{n,j} + \frac{z}{2})X_{n,j} + D^p_{n,j}X_{n,j} + \alpha_{n,j}X^2_{n,j} + \beta_{n,j}X_{n,j}D^t_{n,j} + \gamma_{n,j}(D^t_{n,j})^2$ $x_{n,j} = \frac{1}{2}\delta_{n+1,j}[\epsilon_{n+1,j}X_{n,j} - \phi_{n+1,j}D^t_{n,j}]$ $\alpha_{last,d} = \begin{cases} \frac{1}{q_1} & \text{if } last = 0 \\ \frac{1}{q_2} & \text{if } last = N \\ \frac{1}{2q} & \text{otherwise} \end{cases}, \quad \beta_{last,d} = 1$ | $\alpha_{last,d} = \begin{cases} \lambda_1 + \kappa_1 e^{-\rho v_1} \\ \lambda_2 + \kappa_2 e^{-\rho v_2} \\ \frac{1}{2q} \end{cases}, \quad \beta_{last,d} = \begin{cases} e^{-\rho v_1} & \text{if } last = 0 \\ e^{-\rho v_2} & \text{if } last = N \\ 1 & \text{otherwise} \end{cases}$ α, β and γ as above $\delta = \left(\widetilde{\delta} - \lambda^a + \alpha - \kappa^a e^{-\rho\tau^a}\beta + (\kappa^a)^2 e^{-2\rho\tau^a}\gamma\right)^{-1}$ | $\delta = \left(\alpha - \kappa^b e^{-\rho\tau^b}\beta + (\kappa^b)^2 e^{-2\rho\tau^b}\gamma + \begin{cases} \kappa_1 e^{-\rho v_1} & \text{for } n = 1 \\ \kappa_2 e^{-\rho v_2} & \text{for } n = 0 \\ \frac{1}{2q} - \lambda & \text{otherwise} \end{cases}\right)^{-1}$ $\epsilon = -\lambda^a + 2\alpha - \kappa^a e^{-\rho\tau^a}\beta$ | $\epsilon = -\lambda^b + 2\alpha - \kappa^b e^{-\rho\tau^b}\beta$ $\phi = 1 - e^{-\rho\tau^a}\beta + 2\kappa^a e^{-2\rho\tau^a}\gamma$ | $\phi = -e^{-\rho\tau^b}\beta + 2\kappa^b e^{-2\rho\tau^b}\gamma + \begin{cases} e^{-\rho v_1} & \text{for } n = 1 \\ e^{-\rho v_2} & \text{for } n = 0 \\ 1 & \text{otherwise} \end{cases}$ $\lambda^a_n = \lambda^b_n = \begin{cases} \lambda_1 \\ \lambda_2 \\ \lambda \end{cases}, \quad \kappa^a_n = \kappa^b_n = \begin{cases} \kappa_1 \\ \kappa_2 \\ \kappa \end{cases}, \quad \widetilde{\delta}_n = \begin{cases} \frac{1}{q_1} & \text{for } n = 1 \\ \frac{1}{q_2} & \text{for } n = 0 \\ \frac{1}{2q} & \text{otherwise} \end{cases}$

III

$$C_{n,j} = (S_{n,j} + \tfrac{z}{2})X_{n,j} + D^p_{n,j}X_{n,j} + \alpha_{n,j}X^2_{n,j} + \beta^A_{n,j}X_{n,j}D^A_{n,j} + \beta^B_{n,j}X_{n,j}D^B_{n,j} + \gamma^A_{n,j}(D^A_{n,j})^2 + \gamma^B_{n,j}(D^B_{n,j})^2 + \gamma^{AB}_{n,j}D^A_{n,j}D^B_{n,j}$$

$$x_{n,j} = \tfrac{1}{2}\delta_{n+1,j}[\epsilon_{n+1,j}X_{n,j} - \phi^A_{n+1,j}D^A_{n,j} - \phi^B_{n+1,j}D^B_{n,j}]$$

IIIa

$$\alpha_{last,d} = \begin{cases} \frac{1}{2q_1} & \text{if } last = 0 \\ \frac{1}{2q_2} & \text{if } last = N, \quad \beta^A_{last,d} = 1 \\ \frac{1}{2q} & \text{otherwise} \end{cases}$$

$$\alpha = \alpha' - \tfrac{1}{4}\delta'\epsilon'^2$$

From now on we do not state the case differentiation when it would only be a repeat of the last explicit one.

$$\beta^A = \tfrac{1}{2}\delta'\epsilon'(\phi^A)' + \begin{cases} \frac{1}{2}e^{-\rho v_2}((\beta^A)' + (\beta^B)') & \text{if n=N-1} \\ e^{-\rho(\tau^a)'}((\beta^A)' + (\beta^B)') & \text{if n=0,N} \\ e^{-\rho\tau}(\beta^A)' & \text{otherwise} \end{cases}$$

$$\beta^B = \tfrac{1}{2}\delta'\epsilon'(\phi^B)' + \begin{cases} \frac{1}{2}e^{-\rho v_2}((\beta^A)' + (\beta^B)') \\ 0 \\ e^{-\rho\tau}(\beta^B)' \end{cases}$$

$$\gamma^A = -\tfrac{1}{4}\delta'((\phi^A)')^2 + \begin{cases} \frac{1}{4}e^{-2\rho v_2}((\gamma^A)' + (\gamma^B)' + (\gamma^{AB})') \\ e^{-2\rho(\tau^a)'}((\gamma^A)' + (\gamma^B)' + (\gamma^{AB})') \\ e^{-2\rho\tau}(\gamma^A)' \end{cases}$$

$$\gamma^B = -\tfrac{1}{4}\delta'((\phi^B)')^2 + \begin{cases} \frac{1}{4}e^{-2\rho v_2}((\gamma^A)' + (\gamma^B)' + (\gamma^{AB})') \\ 0 \\ e^{-2\rho\tau}(\gamma^B)' \end{cases}$$

$$\gamma^{AB} = -\tfrac{1}{2}\delta'(\phi^A)'(\phi^B)' + \begin{cases} \frac{1}{2}e^{-2\rho v_2}((\gamma^A)' + (\gamma^B)' + (\gamma^{AB})') \\ 0 \\ e^{-2\rho\tau}(\gamma^{AB})' \end{cases}$$

$$\delta = \begin{cases} (\frac{1}{2q} - \lambda + \alpha - \frac{1}{2}(\kappa - \lambda)e^{-\rho v_2}(\beta^A + \beta^B) + \frac{1}{4}(\kappa - \lambda)^2 e^{-2\rho v_2}(\gamma^A + \gamma^B + \gamma^{AB}))^{-1} & \text{for } n = N \\ (\frac{\tilde{\delta}}{2} - \lambda^a + \alpha - \frac{1}{2}(\kappa^a - \lambda^a)e^{-\rho\tau^a}(\beta^A + \beta^B) + \frac{1}{4}(\kappa^a - \lambda^a)^2 e^{-2\rho\tau^a}(\gamma^A + \gamma^B + \gamma^{AB}))^{-1} & \text{for } n = 0, 1 \\ (\frac{1}{2q} - \lambda + \alpha + e^{-\rho\tau}(-\kappa\beta^A + \lambda\beta^B) + e^{-2\rho\tau}(\kappa^2\gamma^A + \lambda^2\gamma^B - \kappa\lambda\gamma^{AB}))^{-1} & \text{otherwise} \end{cases}$$

$$\epsilon = \begin{cases} 2\alpha - \lambda^a - \frac{1}{2}(\kappa^a - \lambda^a)e^{-\rho\tau^a}(\beta^A + \beta^B) & \text{if n=0,1,N} \\ 2\alpha - \lambda + e^{-\rho\tau}(-\kappa\beta^A + \lambda\beta^B) & \text{otherwise} \end{cases}$$

$$\phi^A = \begin{cases} 1 - \frac{1}{2}e^{-\rho v_2}(\beta^A + \beta^B) + \frac{1}{2}(\kappa - \lambda)e^{-2\rho v_2}(\gamma^A + \gamma^B + \gamma^{AB}) & \text{if n=N} \\ 1 - e^{-\rho\tau^a}(\beta^A + \beta^B) + (\kappa^a - \lambda^a)e^{-2\rho\tau^a}(\gamma^A + \gamma^B + \gamma^{AB}) & \text{if n=0,1} \\ 1 - e^{-\rho\tau}\beta^A + e^{-2\rho\tau}(2\kappa\gamma^A - \lambda\gamma^{AB}) & \text{otherwise} \end{cases}$$

$$\phi^B = \begin{cases} -\frac{1}{2}e^{-\rho v_2}\left(\beta^A + \beta^B\right) + \frac{1}{2}(\kappa - \lambda)e^{-2\rho v_2}\left(\gamma^A + \gamma^B + \gamma^{AB}\right) \\ 0 \\ -e^{-\rho\tau}\beta^B + e^{-2\rho\tau}\left(-2\lambda\gamma^B + \kappa\gamma^{AB}\right) \end{cases}$$

IIIb

$$\alpha_{last,d} = \begin{cases} \lambda_1 + \frac{1}{2}(\kappa_1 - \lambda_1)e^{-\rho v_1} \\ \lambda_2 + \frac{1}{2}(\kappa_2 - \lambda_2)e^{-\rho v_2} \\ \frac{1}{2q} \end{cases} , \quad \beta^A_{last,d} = \begin{cases} e^{-\rho v_1} & \text{if } last = 0 \\ e^{-\rho v_2} & \text{if } last = N \\ 1 & \text{otherwise} \end{cases}$$

$$\alpha = \alpha' - \frac{1}{4}\delta'\epsilon'^2$$

$$\beta^A = \frac{1}{2}\delta'\epsilon'(\phi^A)' + \begin{cases} \frac{1}{2}e^{-\rho\tau}((\beta^A)' + (\beta^B)') & \text{if n=N-1} \\ e^{-\rho(\tau^b)'}((\beta^A)' + (\beta^B)') & \text{if n=0,N} \\ e^{-\rho\tau}(\beta^A)' & \text{otherwise} \end{cases}$$

$$\beta^B = \frac{1}{2}\delta'\epsilon'(\phi^B)') + \begin{cases} \frac{1}{2}e^{-\rho\tau}((\beta^A)' + (\beta^B)') \\ 0 \\ e^{-\rho\tau}(\beta^B)' \end{cases}$$

$$\gamma^A = -\frac{1}{4}\delta'((\phi^A)')^2 + \begin{cases} \frac{1}{4}e^{-2\rho\tau}((\gamma^A)' + (\gamma^B)' + (\gamma^{AB})') \\ e^{-2\rho(\tau^b)'}((\gamma^A)' + (\gamma^B)' + (\gamma^{AB})') \\ e^{-2\rho\tau}(\gamma^A)' \end{cases}$$

$$\gamma^B = -\frac{1}{4}\delta'((\phi^B)')^2 + \begin{cases} \frac{1}{4}e^{-2\rho\tau}((\gamma^A)' + (\gamma^B)' + (\gamma^{AB})') \\ 0 \\ e^{-2\rho\tau}(\gamma^B)' \end{cases}$$

$$\gamma^{AB} = \begin{cases} \frac{1}{2}e^{-2\rho\tau}((\gamma^A)' + (\gamma^B)' + (\gamma^{AB})') \\ 0 \\ e^{-2\rho\tau}(\gamma^{AB})' \end{cases} - \frac{1}{2}\delta'(\phi^A)'(\phi^B)'$$

$$\delta = \begin{cases} (\frac{1}{2q} - \lambda + \alpha - \frac{1}{2}(\kappa - \lambda)e^{-\rho\tau}(\beta^A + \beta^B) + \frac{1}{4}(\kappa - \lambda)^2 e^{-2\rho\tau}(\gamma^A + \gamma^B + \gamma^{AB}))^{-1} & \text{for } n = N \\ (\alpha - \frac{1}{2}(\kappa^b - \lambda^b)e^{-\rho\tau^b}(\beta^A + \beta^B - 1) + \frac{1}{4}(\kappa^b - \lambda^b)^2 e^{-2\rho\tau^b}(\gamma^A + \gamma^B + \gamma^{AB}))^{-1} & \text{for } n = 0, 1 \\ (\frac{1}{2q} - \lambda + \alpha + e^{-\rho\tau}(-\kappa\beta^A + \lambda\beta^B) + e^{-2\rho\tau}(\kappa^2\gamma^A + \lambda^2\gamma^B - \kappa\lambda\gamma^{AB}))^{-1} & \text{otherwise} \end{cases}$$

$$\epsilon = \begin{cases} 2\alpha - \lambda^b - \frac{1}{2}(\kappa^b - \lambda^b)e^{-\rho\tau^b}(\beta^A + \beta^B) & \text{if n=0,1,N} \\ 2\alpha - \lambda + e^{-\rho\tau}(-\kappa\beta^A + \lambda\beta^B) & \text{otherwise} \end{cases}$$

$$\phi^A = \begin{cases} 1 - \frac{1}{2}e^{-\rho\tau}(\beta^A + \beta^B) + \frac{1}{2}(\kappa - \lambda)e^{-2\rho\tau}(\gamma^A + \gamma^B + \gamma^{AB}) & \text{if n=N} \\ -e^{-\rho\tau^b}(\beta^A + \beta^B - 1) + (\kappa^b - \lambda^b)e^{-2\rho\tau^b}(\gamma^A + \gamma^B + \gamma^{AB}) & \text{if n=0,1} \\ 1 - e^{-\rho\tau}\beta^A + e^{-2\rho\tau}(2\kappa\gamma^A - \lambda\gamma^{AB}) & \text{otherwise} \end{cases}$$

$$\phi^B = \begin{cases} -\frac{1}{2}(\kappa - \lambda)e^{-\rho\tau}(\beta^A + \beta^B) + \frac{1}{2}(\kappa - \lambda)e^{-2\rho\tau}(\gamma^A + \gamma^B + \gamma^{AB}) \\ 0 \\ -e^{-\rho\tau}\beta^B + e^{-2\rho\tau}(-2\lambda\gamma^B + \kappa\gamma^{AB}) \end{cases}$$

A.4 Existence of a Lagrange multiplier

In the sequel we translate Theorem 4, page 109 of [10] which guarantees the existence of a Lagrange multiplier and we present it in the form as needed in Proposition 29. To do so we start by stating what we mean by a local extremum.

Definition 38. (Local minimum) Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ a function. A point $x \in U$ will be called a local minimum of f if there is a neighbourhood $V \subset U$ of x such that $f(x) \leq f(y)$ for all $y \in V$.

Theorem 39. *(Lagrange multiplier)*
Let $U \subset \mathbb{R}^n$ be an open set and $\Xi \subset U$ an one-codimensional submanifold with

$$\Xi = \{x \in U | l(x) = 0\}$$

and $l : U \to \mathbb{R}$ continuously differentiable with

$$Rank \left[\frac{\partial}{\partial(x_1, ..., x_n)} l(x) \right] = 1 \quad for\ all \quad x \in \Xi.$$

Furthermore assume that $\widetilde{C}_0 : U \to \mathbb{R}$ is a continuously differentiable function such that $\widetilde{C}_0$ has a local minimum a in Ξ. Then there exists a constant $\nu \in \mathbb{R}$ such that

$$\nabla \widetilde{C}_0(a) = \nu \nabla g(a).$$

Remark 40. In our case we have $U = \mathbb{R}^{N+1}$ and $l(x_0, ..., x_N) = \sum_{n=0}^{N} x_n - X_0$. Then

$$\Xi = \left\{ (x_0, ..., x_N) \in \mathbb{R}^{N+1} \middle| \sum_{n=0}^{N} x_n = X_0 \right\}$$

is an one-codimensional submanifold.

Example	$g(x)$	$\lim_{x\to\infty} g(x)$	$-g'(x)x$
3	$\exp(-x\,(1-e^{-\rho\tau}))$	0	$\leq e^{-1}$
4	$\frac{e^{-\rho\tau}x+10}{x+10}$	$e^{-\rho\tau}$	$\leq \frac{1}{4}\,(1-e^{-\rho\tau})$
5	$\frac{e^{-2\rho\tau}x^2+10}{x^2+10}$	$e^{-2\rho\tau}$	$\leq \frac{1}{2}(1-e^{-2\rho\tau})$

Table 11: h_D is one-to-one for Example 3 to 5.

A.5 Examples satisfy assumptions of Proposition 29 and 33

Let us start with Proposition 29 where h_E has to be one-to-one. Corresponding to Remark 32 and due to the positivity of the optimal strategy we have to show that the function $h_E : \mathbb{R}_+ \to \mathbb{R}_+$ is one-to-one or increasing respectively for Examples 3, 4 and 5. For Example 3 we have

$$h'_E(u) = \frac{e^{-\rho\tau}u + q - e^{-2\rho\tau}(u+q)}{(u+q)\,(e^{-\rho\tau}u+q)},$$

which is obviously positive since all variables are positive. For Examples 4 and 5 we consider the positivity of the function l from (66). In the case of Example 4 we get

$$l(u) = \sqrt{\frac{q}{5}}\left[\sqrt{5q+e^{-\rho\tau}u} - e^{-2\rho\tau}\sqrt{5q+u}\right] > 0$$

and although l for Example 5 gets quite complicated, its positivity can be shown by plotting it for different choices of q and ρ.

We now turn to Proposition 33. According to the last step of the proof of Proposition 33 we only have to check that the function $h_D : \mathbb{R}_+ \to \mathbb{R}_+$ where $h_D(0) = 0$ is one-to-one. In case of Example 0 and 2, h_D is a straight line and a parabola, respectively. For the remaining examples we write

$$h_D(x) = x\frac{1-e^{-2\rho\tau}g(x)}{1-e^{-\rho\tau}g(x)} \quad \text{with} \quad g(x) := \frac{f(e^{-\rho\tau}x)}{f(x)} \quad \text{and} \quad g(0) = 1.$$

Hence, h_D is increasing and therefore one-to-one if

$$(1-e^{-2\rho\tau}g(x))\left(1-e^{-\rho\tau}g(x)\right) > -g'(x)xe^{-\rho\tau}\left(1-e^{-\rho\tau}\right). \tag{79}$$

In case of Example 1 we get $g(x) = \sqrt{\frac{x+1}{e^{-\rho\tau}x+1}}$, which is increasing and satisfies (79). For the Examples 3 to 5 we use that g as stated in Table 11 is decreasing and therefore we get (79) for $x \geq 0$ if

$$e^{\rho\tau} - e^{-\rho\tau} > -g'(x)x. \tag{80}$$

According to Table 11, (80) is satisfied for Example 4 and 5. In case of Example 3, inequality (80) holds if $\rho\tau > 0.2$ which is true for reasonable choices of ρ and τ.

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